Exercise 5

A Sequences

1
Solution

(a) $u_3 = 2u_2 - u_1 + 5$ ◁ substituting $u_1 = a$ and $u_2 = b$

$= 2b - a + 5$

(b) Given $u_{n+2} = 2u_{n+1} - u_n + 5$ ◁ substituting $n = 2$

$u_4 = 2u_3 - u_2 + 5$ ◁ substituting $u_2 = b$ and $u_3 = 2b - a + 5$

$= 2(2b - a + 5) - b + 5$

$= 4b - 2a + 10 - b + 5$

$= 3b - 2a + 15$

Given $u_4 = 3u_2$

$3b - 2a + 15 = 3b$

$2a = 15$

$a = \frac{15}{2}$

2
Solution

(a) Given $u_1 = 2$ and $u_{n+1} = 3u_n - 2$.

$u_2 = 3u_1 - 2 = 3(2) - 2 = 4$

$u_3 = 3u_2 - 2 = 3(4) - 2 = 10$

$u_4 = 3u_3 - 2 = 3(10) - 2 = 28$

(b)(i)

Given $x_n = u_n - 1$

Replace n by $n+1$

$x_{n+1} = u_{n+1} - 1$ .... (A)

Given $u_{n+1} = 3u_n - 2$ .... (B)

Substituting (B) into (A)

$$\begin{aligned} x_{n+1} &= 3u_n - 2 - 1 \\ &= 3u_n - 3 \\ &= 3(u_n - 1) \\ &= 3x_n \quad \text{(Shown) .... (1)} \end{aligned}$$

Substituting $n = 1$ into (1).

$x_2 = 3x_1$

Substituting $n = 2$ into (1)

$$\begin{aligned} x_3 &= 3x_2 \\ &= 3(3x_1) \\ &= 3^2 x_1 \end{aligned}$$

Substituting $n = 3$ into (1)

$$\begin{aligned} x_4 &= 3x_3 \\ &= 3(3^2 x_1) \\ &= 3^3 x_1 \end{aligned}$$

From observation, $x_n = 3^{n-1} x_1$ (Shown)

(ii) Given $x_n = u_n - 1$

Rewrite $u_n = x_n + 1$ ◁ replace $x_n = 3^{n-1} x_1$ (see in **(i)**)

$u_n = 3^{n-1} x_1 + 1$ ◁ substituting $x_n = u_n - 1$ with $n = 1$

$= 3^{n-1}(u_1 - 1) + 1$

3
Solution

(a) Given $u_0 = 5$

and $u_n = 3u_{n-1} - 2,\ n \geq 1$ .... (1)

From (1) : $u_n = 3u_{n-1} - 2$

$= 3(3u_{n-2} - 2) - 2$ ⊲ note: $3u_{n-2}$ is obtained when substituting $n-1$ into $u_n = 3u_{n-1} - 2$

$= 3^2 u_{n-2} - 3(2) - 2$

$= 3^2 u_{n-2} - 2(3+1)$

$= 3^2(3u_{n-3} - 2) - 2(3+1)$ ⊲ note: $3u_{n-3}$ is obtained when substituting $n-2$ into $u_n = 3u_{n-1} - 2$

$= 3^2 3u_{n-3} - 3^2 \times 2 - 2(3+1)$

$= 3^3 u_{n-3} - 18 - 2(3+1)$

$= 3^3 u_{n-3} - 2(9+3+1)$

$= 3^3 u_{n-3} - 2(3^2 + 3^1 + 3^0)$

$\vdots$

$= 3^n u_0 - 2(3^{n-1} + 3^{n-2} + \ldots + 3 + 1)$ ⊲ $u_0 = 5$

$= 3^n(5) - 2(3^0 + 3^1 + 3^2 + +3^{n-2} + 3^{n-1})$

$= 3^n(5) - 2\left[\dfrac{1(3^n - 1)}{3-1}\right]$ ⊲ using GP formula

$= 3^n(5) - (3^n - 1)$

$= 3^n(5) - 3^n + 1$

$= 4(3^n) + 1$ (Shown)

(b)(i)

Given $x_{n+1} = \sqrt{x_n + 5}$.

As $n \to \infty, x_n \to l,\ x_{n+1} \to l$

So $l = \sqrt{(l+5)}$

$l = (l+5)^{\frac{1}{2}}$ ⊲ square both sides

$l^2 = l + 5$ .... (1)

$l^2 - l - 5 = 0$

$l = \dfrac{1 \pm \sqrt{1+20}}{2}$

$= \dfrac{1-\sqrt{21}}{2}$ (rejected since x_n is positive) or $\dfrac{1+\sqrt{21}}{2}$

Hence $l = \dfrac{1+\sqrt{21}}{2}$

(ii) L.H.S

$= (x_{n+1})^2 - l^2$ $\quad \triangleleft$ From (1): $l^2 = l + 5$

$= x_n + 5 - (l + 5)$

$= x_n - l$ $\qquad$ (Proved)

Given $\quad x_n > 1$

$x_n - l > 0$ $\quad \triangleleft$ use the result in **(ii)**: $(x_{n+1})^2 - l^2 = x_n - l$

$(x_{n+1})^2 - l^2 > 0$

$(x_{n+1})^2 > l^2$

$x_{n+1} > l$

When $x_n > l$

$(x_n)^2 - x_n - 5 > 0$

$(x_n)^2 - (x_n + 5) > 0$ $\quad \triangleleft$ use $x_{n+1} = \sqrt{(x_n + 5)} \Rightarrow x_{n+1} = (x_{n+1})^2$

$(x_n)^2 - (x_{n+1})^2 > 0$

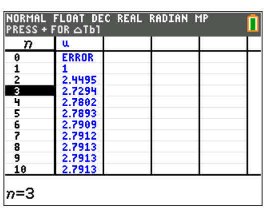
$(x_n)^2 > (x_{n+1})^2$

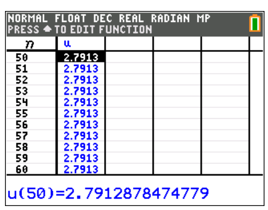
$x_n > x_{n+1}$

$\therefore\ x_n > x_{n+1} > l$ $\quad$ (Shown)

(iii) Given $x_{n+1} = \sqrt{x_n + 5}$.

Using GC





The sequence strictly increases and converges to 2.79. (to 3 sf)

Exercise 5

B Convergence of a sequence or series

4
Solution

(a) As $n \to \infty, n^2 + 1 \to \infty$.

Thus sequence diverges.

(b) By performing long division,

$$u_n = \frac{n+1}{n+2}$$
$$= 1 - \frac{1}{n+2}$$

As $n \to \infty, n+2 \to \infty$. $\therefore \dfrac{1}{n+2} \to 0$

$$\therefore\ 1 - \frac{1}{n+2} = 1 - 0 \to 1$$

The sequence converges to 1.

(c) $u_n = \dfrac{2n^2 + n + 1}{n^2 + 2}$ ◁ divide each term in u_n by highest degree of polyonmial

$$u_n = \frac{\dfrac{2n^2}{n^2} + \dfrac{n}{n^2} + \dfrac{1}{n^2}}{\dfrac{n^2}{n^2} + \dfrac{2}{n^2}}$$
$$= \frac{2 + \dfrac{1}{n} + \dfrac{1}{n^2}}{1 + \dfrac{2}{n^2}}$$

As $n \to \infty, \dfrac{1}{n} \to 0,\ \dfrac{1}{n^2} \to 0,\ \dfrac{2}{n^2} \to 0.$

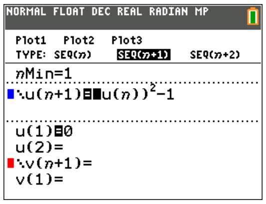
$$\therefore\ \frac{2 + \dfrac{1}{n} + \dfrac{1}{n^2}}{1 + \dfrac{2}{n^2}} \to \frac{2+0+0}{1+0} = 2$$

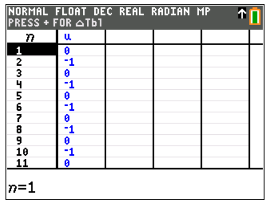
The sequence converges to 2.

5
Solution

(a) Given $u_{n+1} = u_n^2 - 1$

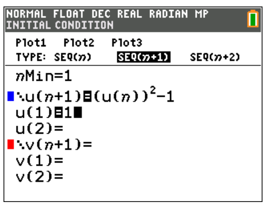
Case when $u_1 = 0$

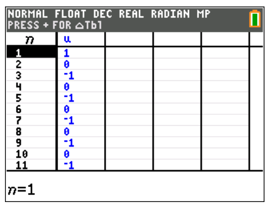




The sequence oscillates from 0 to -1.

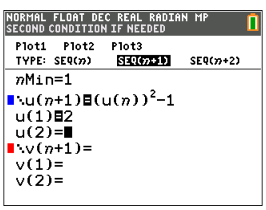
Case when $u_1 = 1$

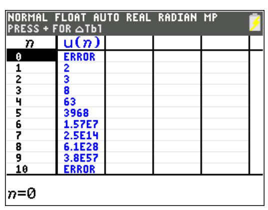




The sequence oscillates from 1 to 0.

Case when $u_1 = 2$





The sequence gets increasingly large (or diverge).

(b) Given$u_{n+1} = u_n^2 - 1$ .... (1)

When $n = 1$,

$$u_2 = u_1^2 - 1 \text{ .... (2)}$$

Given that $u_2 = u_1$ .... (3)

Substituting (3) into (2).

$$u_1 = u_1^2 - 1$$

$$u_1^2 - u_1 - 1 = 0$$

$$u_1 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ the two possible values of u_1 are $\dfrac{1 \pm \sqrt{5}}{2}$.

(c) Substituting $n = 2$ into (1).

$$u_3 = u_2^2 - 1 \text{ .... (4)}$$

From (2): $u_2^2 = (u_1^2 - 1)^2$ .... (5) ⊲ square both sides

Substituting (5) into (4):

$$u_3 = (u_1^2 - 1)^2 - 1$$

Given that U_3=U_1

So $u_1 = u_1^4 - 2u_1^2 + 1$

$\therefore$ $u_1^4 - 2u_1^2 - u_1 = 0$ (Shown)

6
Solution

(a)

Given $u_{n-1} = 0.5u_n + 25$ $\quad\triangleleft$ replace $n-1$ by n

$u_n = 0.5u_{n-1} + 25$

$= 0.5(0.5u_{n-2} + 25) + 25$ $\quad\triangleleft$ note: $0.5u_{n-2}$ is obtained when substituting $n-1$ into $u_{n-1} = 0.5u_n + 25$

$= 0.5(0.5u_{n-2}) + 25(0.5) + 25$

$= 0.5^2 u_{n-2} + 25(1+0.5)$ $\quad\triangleleft$ note: $0.5u_{n-3}$ is obtained when substituting $n-2$ into $u_{n-1} = 0.5u_n + 25$

$= 0.5^2(0.5u_{n-3} + 25) + 25(1+0.5)$

$= 0.5^2(0.5u_{n-3}) + 0.5^2(25) + 25(1+0.5)$

$= 0.5^3 u_{n-3} + 25(1+0.5+0.5^2)$

$= 0.5^3 u_{n-3} + 25(1+0.5+0.5^{3-1})$

$= 0.5^n u_0 + 25(1+0.5+0.5^2+\ldots+0.5^{n-1})$

$= 0.5^n u_0 + 25\left(\dfrac{1-0.5^n}{1-0.5}\right)$

$= 0.5^n u_0 + 50(1-0.5^n)$

$= 0.5^n(u_0 - 50) + 50$ $\quad$ (Shown)

(b) From $0.5^n(u_0-50)+50$

As $n \to \infty$, $0.5^n \to 0$, $(0.5^n)(u_0-50) \to 0$

So $0.5^n(u_0-0.5)+50 \to 50$

$\therefore$ the limiting of u_n is 50.

(c) $u_0 = 50$

Learning point:

From $u_n = 0.5^n(u_0-50)+50$.

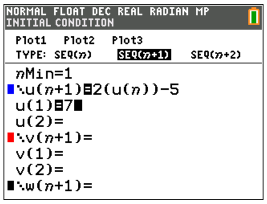
When $u_0 = 50$, $u_n = 0.5^n(50-50)+50 = 50$.

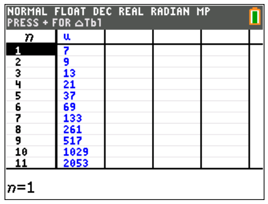
$\therefore$ u_n leads in a constant sequence of 50.

7
Solution

(a)(i)

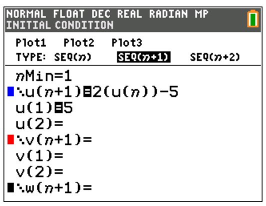
Case A

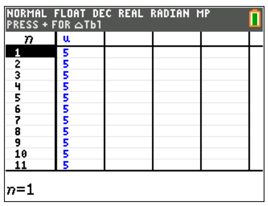




The sequence is increasing as n increases for $n > 0$ and it diverges.

Case B





The sequence leads a constant sequence of 5 as $n > 0$.

(ii) Given $u_{n+1} = 2u_n - 5$

When $n = 1$,

$$u_2 = 2u_1 - 5$$

$$u_2 = 2p - 5 \qquad \triangleleft \text{ given } u_1 = p$$

When $n = 2$,

$$u_3 = 2u_2 - 5 \qquad \triangleleft \text{ substituting } u_2 = 2p - 5$$

$$= 2(2p - 5) - 5$$

$$= 4p - 15$$

When $n = 3$,

$$u_4 = 2u_3 - 5 \qquad \triangleleft \text{ substituting } u_3 = 4p - 15$$

$$u_4 = 2(4p - 15) - 5$$

$$= 8p - 35$$

When $n = 4$,

$$u_5 = 2u_4 - 5 \qquad \triangleleft \text{ substituting } u_4 = 8p - 35$$

$$u_5 = 2(8p - 35) - 5$$

$$= 16p - 75$$

Given $u_5 = 101$,

$$\therefore \quad 16p - 75 = 101$$

$$p = 11$$

(b)(i)

Given $v_{n+2} = v_n + 2v_{n+1} - 7$

When $n = 1$,

$$v_3 = v_1 + 2v_2 - 7 \qquad \triangleleft \text{ given } v_1 = a \text{ and } v_2 = b$$

$$v_3 = a + 2b - 7 \text{ .... (1)}$$

When $n = 2$,

$$v_4 = v_2 + 2v_3 - 7 \qquad \triangleleft \text{ substituting } v_3 = a + 2b - 7$$

$$= b + 2(a + 2b - 7) - 7$$

$$= 2a + 5b - 21 \text{ .... (2)}$$

Given $v_4 = 2v_3$

Equate (1) and (2)

$$\therefore \quad 2a + 5b - 21 = 2(a + 2b - 7)$$

$$2a + 5b - 21 = 2a + 4b - 14$$

$$b = 7$$

(b)(ii)

From (1) and (2):

When $b = 7$, $v_3 = a + 7$ and $v_4 = 2a + 14$

When $n = 3$,

$$v_5 = v_3 + 2v_4 - 7$$

$$= a + 7 + 2(2a + 14) - 7 \qquad \triangleleft \text{ substituting } v_3 = a + 7 \text{ and } v_4 = 2a + 14$$

$$= 5a + 28$$

8
Soluton

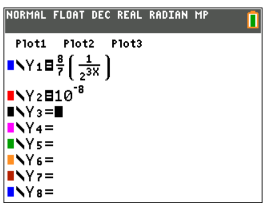
(a) Given $S_n = \frac{8}{7}\left(1 - \frac{1}{2^{3n}}\right)$

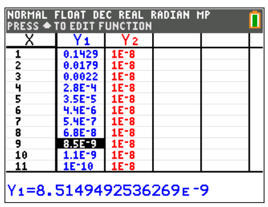
As $n \to \infty$, $\frac{1}{2^{3n}} \to 0$. $\therefore$ $S_n \to \frac{8}{7}$

Since S_n approaches a finite value, it converges.

(b)
$$\left| S_n - S_\infty \right| < 10^{-8}$$
$$\left| \frac{8}{7}\left(1 - \frac{1}{2^{3n}}\right) - \frac{8}{7} \right| < 10^{-8}$$
$$\left| \frac{8}{7} \times \frac{1}{2^{3n}} \right| < 10^{-8}$$

Using GC.





$\therefore$ the smallest value $n = 9$

Exercise 5

C Determine the general term of the series

9
Soluton

(a) Given $S_n = n^3 - 11n^2 + 4n$ and let u_n be nth term.

$$u_n = S_n - S_{n-1},\ n > 1$$

$$= (n^3 - 11n^2 + 4n) - \left[(n-1)^3 - 11(n-1)^2 + 4(n-1)\right]$$

$$= n^3 - 11n^2 + 4n - (n^3 - 3n^2 + 3n - 1) + 11(n^2 - 2n + 1) - 4n + 4$$

$$= 3n^2 - 25n + 16$$

nth term of the series is $3n^2 - 25n + 16$

(b) Given $S_m = S_3$

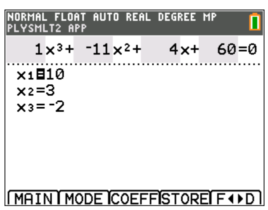
i.e. $m^3 - 11m^2 + 4m = 3^3 - 11(3)^2 + 4(3)$

$m^3 - 11m^2 + 4m = -60$

$m^3 - 11m^2 + 4m + 60 = 0$

Using GC, $m = 10,\ 3,\ -2$

Since $m > 3$, $m = 10$



10
Soluton

(a) $u_n = S_n - S_{n-1}$

$= An^2 + Bn - (A(n-1)^2 + B(n-1))$

$= An^2 + Bn - An^2 + 2An - A - Bn + B$

$= 2An - A + B$

$= A(2n-1) + B$

(b) Given that the tenth term is 48, i.e. $u_{10} = 48$

Substitute $n = 10$ into $u_n = A(2n-1) + B$

$19A + B = 48$

Also given the seventeenth term is 90, i.e. $u_{17} = 90$

Substitute $n = 17$ into $u_n = A(2n-1) + B$.

$33A + B = 90$

Using GC, $A = 3,\ B = -9$

11
Soluton

(a) Given $S_n = \dfrac{6}{13}\left(1 - \dfrac{1}{3^{3n}}\right)$

As $n \to \infty$, $\dfrac{1}{3^{3n}} \to 0.$

$\therefore\ S_\infty = \dfrac{6}{13}$

Given that S_n is within 10^{-8} of the sum to infinity,

i.e. $\left|S_n - S_\infty\right| < 10^{-8}$

$$\left|\frac{6}{13} \times \frac{1}{3^{3n}}\right| < 10^{-8}$$

Using GC, the smallest $n = 6$

(b) $u_n = S_n - S_{n-1}$

$$= \frac{6}{13}\left(1 - \frac{1}{3^{3n}}\right) - \frac{6}{13}\left(1 - \frac{1}{3^{3n-3}}\right)$$

$$= \frac{6}{13} - \frac{6}{13}\left(\frac{1}{3^{3n}}\right) - \frac{6}{13} + \frac{6}{13}\left(\frac{1}{3^{3n-3}}\right)$$

$$= \frac{6}{13}\left(\frac{1}{3^{3n-3}} - \frac{1}{3^{3n}}\right)$$

$$= \frac{6}{13} \times \frac{1}{3^{3n}}\left(\frac{1}{3^{-3}} - 1\right)$$

$$= \frac{6}{13} \times \frac{1}{3^{3n}}(26)$$

$$= \frac{12}{3^{3n}}$$

$$= \frac{3 \times 4}{3^{3n}}$$

$$= \frac{4}{3^{3n-1}}$$

$\therefore$ the formula for u_n is $u_n = \dfrac{4}{3^{3n-1}}$.

Exercise 5

D Arithmetic progression and series

12
Solution

(a)(i) $u_{10} = 8 + 9(-3)$ $\triangleleft$ use $u_n = a + (n-1)d$, where $a = 8$, $n = 10$ and $d = -3$
$= -19$

The 10th term is -19.

(ii) $-37 = 8 + (n-1)(-3)$ $\triangleleft$ use $n\text{th} = a + (n-1)d$, where $a = 8$, $d = -3$, $n\text{th} = -37$

$-45 = (n-1)(-3)$

$15 = n - 1$

$\therefore \; n = 16$

The number of terms in this progression is 16.

(iii) $S_{16} = \frac{16}{2}[2(8) + (16-1)(-3)]$ $\triangleleft$ use $S_n = \frac{n}{2}[2a + (n-1)d]$, where $a = 8$, $d = -3$, $n = 16$
$= -232$

The sum of all the 16 terms is -232.

Alernative Method

$S_{16} = \frac{16}{2}(8 - 37)$ $\triangleleft$ use $S_n = \frac{n}{2}(a + l)$, where $a = 8$, $l = -37$, $n = 16$
$= -232$

(b) Let $a - d, a, a + d$ be the three consecutive terms of an arithmetic progression.
Given that the sum of these three consecutive terms is 21,
so $(a-d) + a + (a+d) = 21$
$a = 7$ ....(1)
Given that the product of 3 consecutive terms is 315,
So $(a-d)(a)(a+d) = 315$ .... (2)
Putting (1) into (2) to find d gives
$(7-d)(7)(7+d) = 315$
$d^2 = 4$
$d = \pm 2$

When $d = 2$ and $a = 7$, the three consecutive terms are $\underline{\underline{5, 7 \text{ and } 9}}$.

When $d = -2$ and $a = 7$, the three consecutive terms are $\underline{\underline{9, 7 \text{ and } 5}}$.

(c) $234 = \frac{18}{2}\left[2a + (30-1)(-0.2)\right]$ ⊲ use $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, where $S_{18} = 234,\ d = -0.2,\ n = 18$

$26 = 2a - 5.8$

$a = 15.9$

The first term is 15.9.

13
Solution

(a) Given $S_4 = 14$,

i.e. $\frac{4}{2}(2a + (4-1)d) = 14$

$$2a + 3d = 7 \text{ .... (1)}$$

Also given he product of the first four terms of the series is 0

i.e. $a(a+d)(a+2d)(a+3d) = 0$ .... (2)

Hence $a = 0$ .... (3)

or $a + d = 0$ .... (4)

or $a + 2d = 0$ .... (5)

or $a + 3d = 0$ .... (6)

From (3): If $a = 0,$ rejected since given that $a < 0$

From (4): If $a + d = 0$

Use GC to solve (1) and (4) simultaneously

$\therefore\ a = -1$ and $d = 7$

From (5): If $a + 2d = 0$

Use GC to solve (1) and (5) simultaneously

$a = 14$ and $d = -7,$ rejected since $a < 0$

From (6): If $a + 3d = 0$

Use GC to solve (1) and (6) simultaneously

$a = 7$ and $d = -\frac{7}{3},$ rejected since $a < 0$

Thus, $a = -7$ and $d = 7$

11th term of the series

$= -7 + (11-1)7$

$= 63$

The 11th term of the series is 63.

(b) $T_{50} = -7 + (50-1)7 = 336$

Number of terms from 11th term to the 50th term inclusive

$= (50-11) + 1$

$= 40$

Sum of the 11th term to the 50th term inclusive

$= \frac{40}{2}(63 + 336)$

$= 7980$

The sum of the 11th term to the 50th term inclusive of this series is 7980.

14
Solution

(a) Given $T_{12} = 52$

i.e. $a + 11d = 52$ .... (1)

Given $S_{18} = 756$

i.e. $\frac{18}{2}(2a + 17d) = 756$

$2a + 17d = 84$ .... (2)

Using GC to solving (1) and (2)

$a = 8$ and $d = 4$.

Given that nth term of the progression is more than 2000,

i.e. $T_n > 2000$

$8 + (n-1)4 > 2000$

$n > 499$

$\therefore$ the least value of n is 500.

(b) $u_n = 1000 + (n-1)(-1.4)$ $\triangleleft$ use $u_n = a + (n-1)d$, where $a = 1000$ and $d = -1.4$

Consider first negative term in the series,

i.e. $u_n < 0$

$1000 + (n-1)(-1.4) < 0$

$n > \frac{1000}{1.4} + 1$

$n > 715.29$

$\therefore$ the first negative term appears when $n = 716$.

First negative term in the series

$= 1000 + (716 - 1)(-1.4)$

$= -1$

Sum of first 20 negative terms

$= \frac{20}{2}[2(-1) + (20-1)(-1.4]$

$= -286$

The sum of the first 20 negative terms of the series is -286.

15
Solution

(a) Given that sum of the first twenty terms of an arithmetic progression is 50,

i.e. $S_{20} = 50$

$$50 = 10(2a + 19d)$$

$$5 = 2a + 19d \text{ .... (1)}$$

Also given that the sum of the next twenty terms is -50.

i.e. $S_{40} - S_{20} = -50$

$$S_{40} = S_{20} - 50$$

$$\frac{40}{2}[2a + (40-1)d] = 50 - 50$$

$$2a + 39d = 0 \text{ .... (2)}$$

Using GC to find a and d :

$$d = -\frac{1}{4} \text{ and } a = \frac{39}{8}.$$

$$\therefore\ S_{100} = \frac{100}{2}\left[2\left(\frac{39}{8}\right) + (100-1)\left(\frac{1}{4}\right)\right]$$

$$= -750$$

The sum of the first hundred terms of the progression is -750.

(b) Given that the sum of the first 5 terms of an arithmetic series is $\frac{1}{3}$ times the sum of its next 5 terms,

i.e. $S_5 = \frac{1}{3}(S_{10} - S_5)$

$$4S_5 = S_{10}$$

$$4\left[\frac{5}{2}(2a + 4(3))\right] = \frac{10}{2}(2a + 9(3))$$

$$20a + 120 = 10a + 135$$

$$a = 1.5$$

The first term of the series is 1.5.

16
Solution

Given that the sum of the first 10 terms in G is twice the 22nd term of the series,

i.e. $\quad S_{10} = 2T_{22}$

$$\frac{10}{2}\left[2a+(10-1)d\right] = 2\left[a+(22-1)\right]$$

$$5(2a+9d) = 2(a+21d)$$

$$8a+3d = 0 \text{ .... (1)}$$

Also given that the 6th term is 37

i.e. $\quad T_6 = 37$

$$a+5d = 37 \text{ .... (2)}$$

Using GC to solve (1) and (2):

$$a = -3,\ d = 8.$$

Arithmetic progression G: $\quad -3,\ 5,\ 13,\ 21,\ 29,\ 37,\ 45,\ 53,\ 61,\ 69,\ 77,\ 85,\ ....$

Arithmetic progression H: $13,\ 37,\ 61,\ 85$

Sum of the first 50 terms of H

$= \frac{50}{2}\left[2(13)+49(24)\right] \quad \triangleleft$ use $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$, where $a = 13,\ d = 24$ and $n = 50$

$= 30\,050$

$\therefore$ the sum of the first 50 terms of H is $30\,050$.

17
Solution

Given that the sum of the first n terms of the progression is 10 000,

i.e. $\quad S_n = 10000$

$$\frac{n}{2}\left[2a+(n-1)10\right]=10000 \qquad \triangleleft \text{ use } S_n=\frac{n}{2}\left[2a+(n-1)d\right], \text{ where } a=a \text{ and } d=10$$

$$a=\frac{10000}{n}-5(n-1)$$

nth term of the progression

$$= a+(n-1)d \qquad \triangleleft \text{ substituting } a=\frac{10000-5n(n-1)}{n} \text{ and } d=10$$

$$=\frac{10000-5n(n-1)}{n}+(n-1)(10)$$

$$=\frac{10000}{n}+5(-n+1+2n-2)$$

$$=\frac{10000}{n}+5(n-1) \qquad \text{(Shown)}$$

Given that the nth term is less than 500

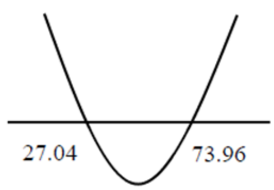
i.e. $$\frac{10000}{n}+5(n-1)<500$$

$$\frac{2000}{n}+(n-1)<100$$

$$2000+n^2-n<100n$$

$$n^2-101n+2000<0$$

$$27.04<n<73.96$$



$\therefore$ the largest possible value of n is 73.

18
Solution

(a) Given $S_n = 12n - n^2$ .... (1)

Substituting $n = 1$ into (1)

$$S_1 = 12(1) - (1)^2$$
$$= 11$$

The first term is 11.

Sum of first 2 terms, $S_2 = u_1 + u_2$ ◁ substituting $n = 2$ into $S_n = 12n - n^2$

$$12(2) - (2)^2 = 11 + u_2$$
$$u_2 = 20 - 11$$
$$u_2 = 9$$

Common difference $= u_2 - u_1$
$= 9 - 11$
$= -2$

The common difference is 11.

(b) Given $T_r = 1 + 4r$ .... (1)

Substituting $n = 1$ into (1)

$$T_1 = 1 + 4(1)$$
$$= 5$$ ◁ first term of AP

Substituting $n = 2$ into (1)

$$T_2 = 1 + 4(2)$$
$$= 9$$ ◁ second term of AP

Common difference $= T_2 - T_1$
$= 9 - 5$
$= 4$

Sum of first n terms, $S_n = \frac{n}{2}[2(5) + (n-1)4]$
$= n(2n + 3)$

The sum of the first n terms of the progression is $n(2n + 3)$.

Alternative Method

$$\sum_{r=1}^{n}(1 + 4r) = \sum_{r=1}^{n} 1 + 4\sum_{r=1}^{n} r$$
$$= n + 4\left[\frac{n}{2}(n+1)\right]$$
$$= n(2n + 3)$$

(c) Given $(1+x^2)(1+x)^n = a+bx+cx^2+\ldots$

L.H.S

$=(1+x^2)(1+x)^n$ ⊲ applying binomial expansion for $(1+x)^n$

$=(1+x^2)(1+nx+\dfrac{n(n-1)x^2}{2}+\ldots)$

$=(1+x^2)(1+nx+\dfrac{n(n-1)x^2}{2}+\ldots)$

$=1+nx+\dfrac{n(n-1)x^2}{2}+x^2\ldots$ .... (1)

Comparing (1) and $a+bx+cx^2+\ldots$

$a=1$ .... (2)

$n=b$ .... (3)

$\dfrac{n(n-1)}{2}+1=c$ .... (4)

Given that a, b, and c are consecutive terms of a AP,

so $b-a=c-b$ .... (5) ⊲ common difference

Substituting (2), (3) and (4) into (5)

$$n-1=\frac{n(n-1)}{2}+1-n$$

$n^2-5n+4=0$

Using GC, $n=1$ or $n=4$

The two possible values of n are 1 and 4.

Exercise 5

E Geometric progression and series

19
Solution

(a)(i)

$$u_{10} = 2(-3)^{10-1} \quad \triangleleft \text{ use } u_n = ar^{n-1}, \text{ where } a = 2,\ r = -3 \text{ and } n = 10$$
$$= -39366$$

The 10th term is -39366.

(ii) $-354\,294 = 2(-3)^{n-1} \quad \triangleleft$ use $u_n = ar^{n-1}$, where $a = 2,\ r = -3$ and $u_n = -354\,294$

$$-177147 = (-3)^{n-1}$$
$$-3^{11} = (-3)^{n-1}$$
$$11 = n - 1$$
$$n = 12$$

There are 12 terms in this progression.

(iii) $S_{12} = \dfrac{2[(-3)^{12} - 1]}{-3-1} \quad \triangleleft$ use $S_n = \dfrac{a(r^n - 1)}{r-1}$, where $a = 2,\ r = -3$ and $n = 12$

$$= -265720$$

The sum of 12 terms is -265720.

(b) Given that the fifth term of a geometric progression is 1

i.e. $u_5 = 1$

$$1 = a\left(\frac{1}{2}\right)^{5-1} \quad \triangleleft \text{ use } u_n = ar^{n-1}, \text{ where } r = \frac{1}{2} \text{ and } u_5 = 1$$
$$a = 16$$

The first term is 16.

$$n\text{th term} = ar^{n-1} \quad \triangleleft\ a = 16 \text{ and } r = \frac{1}{2}$$
$$= 16\left(\frac{1}{2}\right)^{n-1}$$

The nth term is $16\left(\dfrac{1}{2}\right)^{n-1}$.

(c) Given that the sum of the first 16 terms is 262140,

i.e. $S_{16} = 262140$

$$262140 = \frac{4[(r)^{16} - 1]}{r - 1} \qquad \triangleleft \text{ use } S_n = \frac{a(r^n - 1)}{r - 1}, \text{ where } a = 4,\ n = 16,\ S_{16} = 262140$$

$$65535 = \frac{r^{16} - 1}{r - 1}$$

$$0 = r^{16} - 65535r + 65536$$

Using GC, $r = 2$

The common ratio of this progression is 2.

20
Solution

Given that the third term of a geometric progression is 36,
i.e. $u_3 = 36$.

$ar^2 = 36$.............................. (1)

Also given that the sixth term is 121.5,
i.e. $u_3 = 36$.

$ar^5 = 121.5$..........................(2)

Taking $\frac{(2)}{(1)}$:

$r^3 = 3.375$

$\therefore \quad r = 1.5$ ....(3)

Substituting (3) into (1) to find a.

$$\therefore \quad a = \frac{36}{1.5^2}$$
$$= 16$$

Sum of the first eight terms, $S_8 = \frac{16(1-1.5^8)}{1-1.5}$
$= 788.125$

The first term is 16, the common ratio is 1.5 and the sum of the first eight terms is 788.125.

21
Solution

(a) Let the three consecutive terms be $\frac{a}{r},\ a,\ ar$.

Given that three consecutive terms of a geometric progression have a product of 343,

i.e. $\frac{a}{r}(a)(ar) = 343$.............................. (1)

$$a^3 = 343$$

$\therefore$ $a = 7$ .... (2)

Also given that the sum of three consecutive terms is 24.5,

i.e. $\frac{a}{r} + a + ar = 24.5$............................ (3)

Substituting into (2) into (3)

$$\therefore \quad \frac{7}{r} + 7 + 7r = \frac{49}{2}$$

$$2r^2 - 5r + 2 = 0$$

$$(2r-1)(r-2) = 0$$

$$r = \frac{1}{2} \text{ or } 2$$

When $r = \frac{1}{2}$ and $a = 7,$ the three numbers are $14,\ 7,\ \frac{7}{2}$.

When $r = 2$ and $a = 7,$ tThe three numbers are $\frac{7}{2},\ 7,\ 14$.

(b) Let the common ratio be r.

The sum of the first 10 terms, $S_{10} = \frac{8(r^{10}-1)}{r-1}$.

The sum of the reciprocal of these ten terms $= \dfrac{\frac{1}{8}\left(1-\frac{1}{r^{10}}\right)}{1-\frac{1}{r}}$

$$= \frac{r^{10}-1}{8r^9(r-1)}$$

Given that the sum of its first ten terms is $\frac{1}{8}$ of the sum of the reciprocal of these terms,

i.e. $\frac{8(r^{10}-1)}{r-1} = \frac{1}{8}\left(\frac{r^{10}-1}{8r^9(r-1)}\right)$, where $r \neq 1$.

$$r^9 = \frac{1}{64(8)}$$

$\therefore$ $r = 0.5$

The common ratio of this progression is 0.5.

22
Solution

(a) Given that the sum of the first four terms is 0,
i.e. $S_4 = 0$.

$\therefore\ f + fr + fr^2 + fr^3 = 0$ .... (1)

$$1 + r + r^2 + r^3 = 0 \quad (\because f \neq 0)$$

$$(r+1)(r^2+1) = 0$$

$$r = -1 \text{ or } r^2 = -1 \quad \text{(No solution)}$$

Substituting $r = -1$ into (1).

$$f + f(-1) + f(-1)^2 + f(-1)^3 = 0 \text{ .... (2)}$$

From (2): we not that f can take any real values to satisfy (2).

Hence, the possible values of f is $f \in \mathbb{R}$, except $f = 0$.

Alternative Method

Since $S_4 = 0$, hence $r < 0$.

$$\therefore \quad \frac{f(1-r^4)}{1-r} = 0$$

Since $f \neq 0$,

$$1 - r^4 = 0$$

$$\therefore \quad r = -1 \quad \text{or } r = 1 \text{ (Rejected since } r < 0)$$

Hence, the possible values of f is $f \in \mathbb{R}$, except $f = 0$.

The answer can also be expressed as $f \in \mathbb{R} \setminus \{0\}, r = -1$.

Sum of first n terms

$$= \frac{f(1-(-1)^n)}{1-(-1)}$$

$$= \frac{f}{2}\left[1-(-1)^n\right]$$

When n is odd, sum of first n terms $= \frac{f}{2}\left[1-(-1)\right]$

$$= \frac{f}{2}(2n)$$

$$= f$$

When n is even, sum of first n terms $= \frac{f}{2}\left[1-(1)\right]$

$$= 0$$

The answer can also be expressed as

$$S_n = \begin{cases} f, \text{if } n \text{ is odd} \\ 0, \text{if } n \text{ is even} \end{cases}$$

(b) Sum of first k terms $= \dfrac{q(r^k - 1)}{r - 1}$

Sum of last k terms = Sum of first n terms − Sum of first $(n-k)$ terms

$$= \frac{q(r^n - 1)}{r - 1} - \frac{q(r^{n-k} - 1)}{r - 1}$$

$$= \frac{q}{r-1}(r^n - 1 - r^{n-k} + 1)$$

$$= \frac{q}{r-1}(r^n - r^{n-k})$$

$$= \frac{q}{r-1} r^{n-k}(r^n - 1)$$

Difference required

$$= \left| \frac{q(r^k - 1)}{r - 1} - \frac{q}{r-1} r^{n-k}(r^k - 1) \right| \qquad \triangleleft \text{ take out common factor, } \frac{q}{r-1}$$

$$= \left| \frac{q}{r-1}\left[(r^k - 1) - r^{n-k}(r^k - 1) \right] \right|$$

$$= \left| \frac{q}{r-1}(r^k - 1)(1 - r^{n-k}) \right|$$

23
Solution

(a)(i)
Let r be the common ratio.
Given that the 5th term is 1.6384,
i.e. $u_5 = 1.6384$

$$4r^{5-1} = 1.6384 \quad \triangleleft \text{ use } u_n = ar^{n-1}, \text{ where } a = 4,\ r = 5 \text{ and } n = 5$$

$$r = \pm\sqrt[4]{\frac{1.6384}{4}}$$

$$= 0.8 \quad \text{since } r > 0$$

Sum to infinity
$= \dfrac{4}{1-0.8}$
$= 20$

The sum to infinity of this series is 20.

(ii)
Given that the sum of the first n terms is greater than 19.6,
i.e. $S_n > 19.6$

$$\frac{4(1-0.8^n)}{1-0.8} > 19.6$$

$$1-0.8^n > \frac{19.6 \times 0.2}{4}$$

$$1-0.8^n > 0.98$$

$$1-0.98 > 0.8^n$$

i.e. $0.8^n < 0.02$ (shown) .... (1)

From (1): $0.8^n < 0.02 \quad \triangleleft$ take ln on both sides

$$n \ln 0.8 < \ln 0.02$$

$$n > \frac{\ln 0.02}{\ln 0.8}$$

$$> 17.531 \text{ (to 5s.f.)}$$

Hence, the smallest possible value of n is 18.

(b)(i) $S_\infty = \dfrac{\sin\theta}{1-(-\cos\theta)}$ $\quad\triangleleft$ use $S_\infty = \dfrac{a}{1-r}$, where $a=\sin\theta$ and $r=-\cos\theta$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}, \text{ where } k=\frac{1}{2}$$

$\therefore\ k = \dfrac{1}{2}$

(ii) Given that $\theta = \dfrac{\pi}{3}$, $a = \sin\dfrac{\pi}{3}$ and $r = -\cos\dfrac{\pi}{3}$.

$$S_7 = \frac{\sin\frac{\pi}{3}\left[1-\left(-\cos\frac{\pi}{3}\right)^7\right]}{1-\left(-\cos\frac{\pi}{3}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left[1-\left(-\frac{1}{2}\right)^7\right]}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left[1-\left(-\frac{1}{128}\right)\right]}{\frac{3}{2}}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{129}{128}\right)$$

$$= \frac{43\sqrt{3}}{128}$$

The sum of the first seven terms of this series is $\dfrac{43\sqrt{3}}{128}$.

24
Solution

(a)

Sum to infinify of geometric progression G is $S = \dfrac{a}{1-r}$

Given also that sum to infinify of even-numbered terms of geometric progression $G,$ is $-\dfrac{1}{2}S.$

$$\text{sum to infinify of } G = \text{Sum to infinify of odd-numbered terms} + \text{sum to infinify of even-numbered terms}$$

$$S = \text{Sum to infinify of odd-numbered terms} + \left(-\frac{1}{2}S\right)$$

$$\frac{3}{2}S = \text{Sum to infinify of odd-numbered terms}$$

$$\frac{3}{2}S = a + ar^2 + ar^4 + \ldots$$

$$\frac{3}{2}\left(\frac{a}{1-r}\right) = \frac{a}{1-r^2}$$

$$\frac{3}{2}\left(\frac{a}{1-r}\right) = \frac{a}{(1-r)(1+r)}$$

$$\frac{3}{2} = \frac{1}{1+r}$$

$$3 + 3r = 2$$

$$r = -\frac{1}{3}$$

The value of r is $-\dfrac{1}{3}.$

Given that the third term of G is 2,

i.e. $ar^2 = 2 \quad \triangleleft\, r = -\dfrac{1}{3}$ (from **(a)**)

$$a\left(-\frac{1}{3}\right)^2 = 2$$

$$a = 18$$

Sum to infinify of odd-numbered terms of geometric progression G

$= a + ar^2 + ar^4 + \ldots$

$= \dfrac{a}{1-r^2}$

$= \dfrac{18}{1-\left(\frac{1}{3}\right)^2}$

$= \dfrac{81}{4}$ (Shown)

(b) Sum to infinity of H

$$= 18 + \left|18\left(-\frac{1}{3}\right)\right| + \left|18\left(-\frac{1}{3}\right)^2\right| + \left|18\left(-\frac{1}{3}\right)^3\right| + \ldots$$

$$= 18 + 18\left(\frac{1}{3}\right) + 18\left(\frac{1}{3}\right)^2$$

$$= \frac{18}{1-\frac{1}{3}}$$

$$= 27$$

Sum to infinify of geometric progression G

$$S = \frac{18}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{27}{2}$$

$$2S = 27$$

$\therefore$ sum to infinity of $H = 2S$ (Shown)

25
Solution

Sum to infinity of the even-numbered terms of G

$$ar + ar^3 + ar^5 + ... = 78$$

$$\frac{ar}{1-r^2} = 78$$

$$a = \frac{78(1-r^2)}{r} \text{ .... (1)}$$

Sum to infinity of every third-term of G

$$ar^2 + ar^5 + ar^8 + ... = 24$$

$$\frac{ar^2}{1-r^3} = 24$$

$$a = \frac{24(1-r^3)}{r^2} \text{ .... (2)}$$

Equating (1) and (2)

$$\frac{78(1-r^2)}{r} = \frac{24(1-r^3)}{r^2}$$

$$13r(1-r^2) = 4(1-r^3)$$

$$9r^3 - 13r + 4 = 0$$

By GC, $r = -\frac{4}{3}, \frac{1}{3}, 1$

Given that geometric progression G is convergence, then $|r| < 1$.

$\therefore\ r = \frac{1}{3},\ r = -\frac{4}{3}$ (Rejected) and $r = 1$ (Rejected)

Substitute $r = \frac{1}{3}$ into (1).

$$78 = \frac{a\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}$$

$$a = 208$$

26
Solution

(a) Let T_n be the general term of the series

$$T_n = S_n - S_{n-1}$$
$$= \left[1-(\ln x)^n\right]-\left[1-(\ln x)^{n-1}\right]$$
$$= (\ln x)^{n-1} - (\ln x)^n$$
$$= (\ln x)^{n-1}(1-\ln x)$$

$$\frac{T_{n+1}}{T_n} = \frac{(\ln x)^n(1-\ln x)}{(\ln x)^{n-1}(1-\ln x)}$$
$$= \frac{1}{(\ln x)^{-1}}$$
$$= \ln x$$

Since $\frac{T_{n+1}}{T_n} = \ln x$ is a constant and is independent of n, hence it is a geometric series. (Shown)

The common ratio, $r = \ln x$

(b) For the sum to infinity to exist, i.e. $|r| < 1$

So, $|\ln x| < 1$

$$-1 < \ln x < 1$$
$$\frac{1}{\mathrm{e}} < x < \mathrm{e},\ x \neq 1$$

The range of values of x is $\frac{1}{\mathrm{e}} < x < \mathrm{e}$.

(c) As $n \to \infty$, $(\ln x)^n \to 0$, $S_n \to 1-0 = 1$.

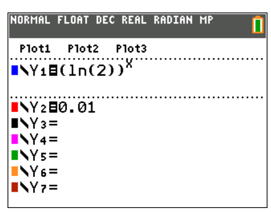
Sum to infinity, $S_\infty = 1$

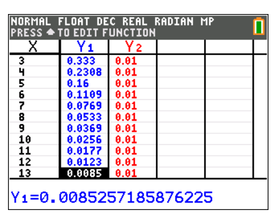
Given that S_n lies within 1% of the sum of infinity,

$\therefore \quad |S_\infty - S_n| < 0.01\,(S_\infty)$

$$\left|1-\left[1-(\ln 2)^n\right]\right| < 0.01(1)$$
$$(\ln 2)^n < 0.01$$

From GC, the least $n = 13$.





Alternative Method

$$(\ln 2)^n < 0.01$$
$$n\ln(\ln 2) < \ln 0.01$$
$$n > \frac{\ln 0.01}{\ln(\ln 2)}$$
$$> 12.6 \text{ (3 s.f.)}$$

$\therefore$ the least $n = 13$

27
Solution

(a) Given $S_n = \dfrac{a^n}{2^{n-1}} - 2$

$S_{n-1} = \dfrac{a^{n-1}}{2^{n-2}} - 2$ ◁ replace n with $n-1$ in $S_n = \dfrac{a^n}{2^{n-1}} - 2$

nth term of the series, T_n

$= S_n - S_{n-1}$

$= \left(\dfrac{a^n}{2^{n-1}} - 2\right) - \left(\dfrac{a^{n-1}}{2^{n-2}} - 2\right)$

$= \dfrac{a^n}{2^{n-1}} - \dfrac{a^{n-1}}{2^{n-2}}$

$= \dfrac{a^n a^{-1}}{a^{-1} \times 2^{n-1}} - \dfrac{a^{n-1}}{2^{n-1} \times 2^{-1}}$

$= \dfrac{a \times a^{n-1}}{2^{n-1}} - \dfrac{a^{n-1}}{2^{n-1} \times 2^{-1}}$

$= \left(\dfrac{a}{2}\right)^{n-1} \left(a - \dfrac{1}{2^{-1}}\right)$

$= \left(\dfrac{a}{2}\right)^{n-1} (a-2)$ (shown)

$\dfrac{T_n}{T_{n-1}} = \dfrac{a-2\left(\dfrac{a}{2}\right)^{n-1}}{a-2\left(\dfrac{a}{2}\right)^{n-2}}$ ◁ replace $n-1$ with n in $T_n = \left(\dfrac{a}{2}\right)^{n-1} (a-2)$ to obtain T_{n-1}

$= \dfrac{a}{2}$

Since $\dfrac{T_{n+1}}{T_n} = \dfrac{a}{2}$ is a constant and is independent of n, hence it is a geometric series. (Shown)

(b) For the sum to infinity to exist, $|r| < 1$

$\therefore \quad \left|\dfrac{a}{2}\right| < 1$

$|a| < 2$

Hence $-2 < a < 2,\ a \neq 0$.

(c) $T_1 = -1,\ S_n = \dfrac{1}{2^{n-1}} - 2,$ Common ratio $= \dfrac{1}{2}$

$$S_\infty = \frac{-1}{1-\frac{1}{2}} = -2$$

Given that S_n to be within ± 0.2 of the value of the sum to infinity,

i.e. $\left|S_n - S_\infty\right| < 0.2$

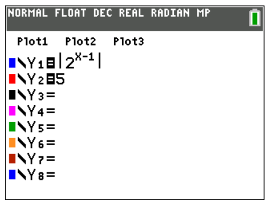
$$\left|\frac{1}{2^{n-1}} - 2 - (-2)\right| < 0.2$$

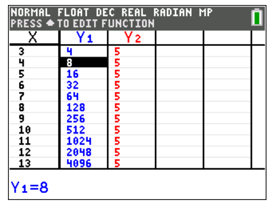
$$\left|\frac{1}{2^{n-1}}\right| < 0.2$$

$$\left|2^{n-1}\right| > 5$$

Using GC, $n = 4$

$\therefore$ the least value of n is 4.





Alternative Method

$$2^{n-1} > 5$$

$$n - 1 > \frac{\ln 5}{\ln 2}$$

$$n > 3.32$$

$$n \geq 4$$

$\therefore$ the least $n = 4$

F Arithmetic and Geometric progression combination

28
Solution

(a) Given that the sum of the first 13 terms of the geometric progression is 156,

i.e. $S_{13} = 156$.

$$\frac{3(1-r^{13})}{1-r} = 156 \qquad \triangleleft \text{ use sum formula of GP: } \frac{a(1-r^n)}{1-r}, \text{ where } n = 13 \text{ and } a = 3$$

$$1-r^{13} = 52-52r$$

$$r^{13} - 52r + 51 = 0 \quad \text{(Shown)}$$

If $r = 1$, $S_{13} = \underbrace{3+3+\ldots+3}_{13\text{ times}} = 39 \neq 156$

Using GC:

$r = -1.4511$ or $r = 1$ (NA) or 1.2100

The possible values of the common ratio are -1.45 or 1.21.

(b) Since $r > 0$, $r = 1.2100$

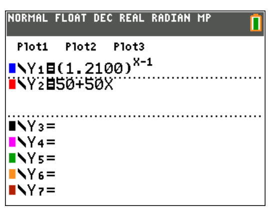
$$3(1.2100)^{n-1} > 100\left(3+(n-1)\left(\frac{3}{2}\right)\right)$$

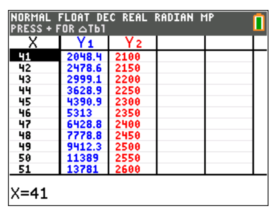
$$(1.2100)^{n-1} > 100 + 50n - 50$$

$$(1.2100)^{n-1} > 50 + 50n$$

Using GC,

n	$(1.2100)^{n-1}$	$50+50n$
41	2048.4	2100
42	2478.6	2150





The smallest value of n is 42.

29
Solution

Let the general term of the geometric series be u_n and the sum have first n term of the arithmetic series be S_n.

Given that the *k*th term of the geometric series is equal to the sum of the first 64 terms of the arithmetic series,

i.e. $u_k = S_{64}$

$$a(2)^{k-1} = \frac{64}{2}\left[2a+(64-1)(2a)\right]$$

$$a(2)^{k-1} = 32(128a)$$

$$2^{k-1} = 2^5(2^7) \quad (\text{Since } a \neq 0)$$

$$k-1 = 12$$

$$k = 13$$

The value of k is 13

30
Solution

(a) $x,\ y,\ z$ are the first three terms of a geometric progression.
The common ratio:

$$\frac{y}{x} = \frac{z}{y}$$

$$\therefore \quad y^2 = xz \text{ .... (1)} \quad \text{(Shown)}$$

$x,\ z,\ y$ are the first three terms of an arithmetic progression.
The common difference:

$$y - z = z - x$$
$$2z = y + x$$
$$z = \frac{y+x}{2} \text{ .... (2)} \quad \text{(Shown)}$$

(b) Substituting (1) into (2)

$$y^2 = x\left(\frac{y+x}{2}\right)$$
$$2y^2 = xy + x^2$$
$$2\left(\frac{y}{x}\right)^2 = \frac{y}{x} + 1$$
$$\left[\left(\frac{2y}{x}\right)+1\right]\left[\frac{y}{x}-1\right] = 0 \quad \text{(Shown)}$$

(c) From (2): Solve the equation $\left[\left(\frac{2y}{x}\right)+1\right]\left[\frac{y}{x}-1\right] = 0$

$\therefore \quad \frac{y}{x} = -\frac{1}{2}$ or $\frac{y}{x} = 1$ (Rejected, $\because$ ratio $\neq$ 1)

Hence, the common ratio is $-\frac{1}{2}$.

Sum to infinity of the geometric progression

$= \dfrac{x}{1-\left(-\frac{1}{2}\right)}$ ◁ use $S_\infty = \dfrac{a}{1-r}$, where $a = x$ and $r = -\dfrac{1}{2}$

$= \dfrac{2}{3}x$ (Shown)

31
Solution

(a)
Let the general term of geometric sequence be u_n and the general term of arithmetic sequence be T_n.

Geometric sequence : $u_1 = a,\ u_2 = ar,\ u_3 = ar^2$
Arithmetic sequence : $T_1 = a,\ T_4 = a+3d,\ T_6 = a+5d$

Given that the first terms of the geometric series is equal to the first of the arithmetic series.
i.e. $u_1 = T_1$
$u_1 = a$

The second term of the geometric series is equal to the fourth term of the arithmetic series.
i.e. $u_2 = T_4$
$ar = a+3d$
$\dfrac{ar-a}{3} = d$ .... (1)

The thrid term of the geometric series is equal to the sixth terms of the arithmetic series.
i.e. $u_3 = T_6$
$ar^2 = a+5d$ .... (2)

Substituting (1) into (2)

$$ar^2 = a + 5\left(\frac{ar-a}{3}\right)$$

$$ar^2 - a = 5\left(\frac{ar-a}{3}\right)$$

$$5(ar-a) = 3(ar^2 - a)$$

$$5ar - 5a = 3ar^2 - 3a$$

$$3ar^2 - 5ar + 2a = 0$$

$3r^2 - 5r + 2 = 0$, where $a \neq 0$ (Shown)

(b) From **(a)**: $3r^2 - 5r + 2 = 0$
$(3r-2)(r-1) = 0$
$r = \dfrac{2}{3}$ or $r = 1$

When $r = 1$, substituting into (1)

$$\frac{a(1)-a}{3} = d$$

$$d = 0$$

Since d is non-zero, so $r = 1$ is rejected.

For geometric series to convergence, $|r| < 1$. Since $r = \dfrac{2}{3}$

Hence geometric series is convergent. (Deduced)

Sum to infinity of the geometric series, $S_\infty = \dfrac{a}{1-\frac{2}{3}}$ $\quad \triangleleft$ where $r = \dfrac{2}{3}$

$$= 3a$$

(c) Substituting $r = \dfrac{2}{3}$ into (1): $d = \dfrac{a\left(\frac{2}{3}\right) - a}{3}$

$$= -\frac{1}{9}a$$

The sum of the first n terms of the arithmetic series exceeds $4a$, i.e. $S > 4a$.

$$\frac{n}{2}\left[2a + (n-1)d\right] > 4a$$

$$\frac{n}{2}\left[2a + (n-1)\left(-\frac{1}{9}a\right)\right] > 4a$$

$$n(2(9) - (n-1)) > 4(2)(9)$$

$$n(19-n) > 72$$

$$n^2 - 19n + 72 < 0$$

Using GC : $5.22 < n < 13.7$

The set of possible values of n is $\{n \in \mathbb{Z} : 6 \le n \le 13\}$.

32
Solution

(a)(i)

Let a and d be the lst term and common difference respectively.
Arithmetic propression of even-numbered terms:

$$a+d,\ a+2d,\ a+4d,\ ....,$$

First term $= a+d$
Common difference $= 2d$
Number of terms $= 444$

Given sum of all even-numbered terms = 408480,

i.e. $\frac{444}{2}\{2(a+d)+(444-1)2d\} = 408480$

$$2a+888d = 1840$$

$$a+444d = 920 \text{ .... (1)}$$

Also given that the 1st term, 9 th term, and the 21st term of the arithmetic progression are three consecutive terms of a geometric progression, i.e. $\frac{u_9}{u_1} = \frac{u_{21}}{u_9}$.

$$\therefore \quad \frac{a+8d}{a} = \frac{a+20d}{a+8d}$$

$$(a+8d)^2 = a(a+20d)$$

$$a^2+16ad+64d^2 = a^2+20ad$$

$$4ad = 64d^2$$

$$a = 16d \text{ .... (2)}$$

Substituting (2) into (1)

$$(16d)+444d = 920$$

$$460d = 920$$

$$d = 2$$

Substituting $d = 2$ into (1)

$$a = 16\times 2$$

$$= 32$$

$\therefore$ the first term is 32 and the common difference is 2.

(b)(i)

Let the first term and the common ratio of the geometric progression be a and r respectively.

Since the GP is convergent, $|r| < 1$.

Given a, ar, ar^3 are consecutive terms of an arithmetic progression.

$$ar^3 - ar = ar - a$$
$$r^3 - r = r - 1$$
$$r(r-1)(r+1) = r - 1$$
$$r(r+1) = 1 \text{ (since } |r|<1,\ r \neq 1 \text{ and so } r - 1 \neq 0)$$

$$\therefore\ r^2 + r - 1 = 0$$
$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$
$$r = \frac{-1+\sqrt{5}}{2} \text{ or } r = \frac{-1-\sqrt{5}}{2} \text{ (reject since } |r| < 1)$$

(b)(ii)

Sum to infinity, $S_\infty = \dfrac{a}{1-r}$

Sum of the first 5 terms, $S_5 = \dfrac{a(1-r^5)}{1-r}$

$$= \frac{a}{1-r}(1-r^5)$$
$$= S_\infty \left[1 - \left(\frac{-1+\sqrt{5}}{2}\right)^5\right]$$
$$= 0.910 S_\infty > 0.9 S_\infty \quad \text{(Shown)}$$

Exercise 5

G Applications I (Basics)

33
Solution

Production in 2014 $= 5000$.
Production goal $= 1.3(5000) = 6500$
Given that 3.5% of the total cartons of milk will be discarded every year, therefore there is only 96.5% cartons of milk remaining to be sold.
Let the production in 2014 be the first term, $a = 5000$

Plan A (follows Arithmetic Progression)

$$0.965(5000+(n-1)105) \geq 6500 \quad \triangleleft \text{ use } n\text{th term of AP: } a+(n-1)d, \text{ where } a = 5000 \text{ and } d = 105$$
$$n-1 \geq 16.53$$
$$n \geq 17.53$$

$\therefore\ n = 18$ years
If the factory adopts Plan A, it will take 18 years first reach to produce 6 500 cartons of milk.

Plan B (follows Geometric Progression)

$$0.965(5000(1.02)^{n-1}) \geq 6500 \quad \triangleleft \text{ use } n\text{th term of GP: } ar^{n-1}, \text{ where } a = 5000 \text{ and } r = 1.02$$
$$(1.02)^{n-1} \geq 1.34715$$
$$n-1 \geq 15.048$$
$$n \geq 16.048$$

$\therefore\ n = 17$ years
If the factory adopts Plan B, it will take 17 years to first reach to produce 6 500 cartons of milk.

$\therefore$ **Plan B** is better. It will achieve in year 2030.

34
Solution

(a) Method A (follows Arithmetic Progression)
There are 20 rectangular planks, so number of terms, $n = 20$
The shortest plank has length 65 cm, let the shortest plank be the first term, $a = 65$
The longest plank has length 350 cm, let the longest plank be the 20th term, $u_{20} = 350$

(i) $u_{20} = a + (n-1)d$ ◁ use nth term of AP: $a + (n-1)d$

$350 = 65 + 19d$

$d = 15$

The value of d is 15.

(ii) Total length of all the planks, S_{20}

$= \frac{20}{2}(65 + 350)$ ◁ use sum of AP: $\frac{n}{2}(a + l)$, where $a = 65$, $l = 350$ (last term) and $n = 20$

$= 4150 \text{ cm}$

(b) Method B (follows Geometric Progression)
Length of long plank is 1.05 m (1050 cm), so $S_n = 1050$
The shortest plank has length 128 cm, so the first term, $a = 128$
Each successive plank is $\frac{8}{9}$ times of the preceding plank, so the common ratio, $r = \frac{8}{9}$.

(i) Given that the length of the nth piece of plank is p cm,
i.e. $p = u_n$

$= 128\left(\frac{8}{9}\right)^{n-1}$ ◁ use nth term of GP: ar^{n-1}, where $a = 128$ and $r = \frac{8}{9}$

$\ln p = \ln\left[128\left(\frac{8}{9}\right)^{n-1}\right]$ ◁ introduce ln both sides

$= \ln 128 + (n-1)\ln\left(\frac{8}{9}\right)$

$= \ln 2^7 + (n-1)\ln 2^3 - (n-1)\ln 3^2$

$= 7\ln 2 + 3(n-1)\ln 2 - 2(n-1)\ln 3$

$= 7\ln 2 + 3n\ln 2 - 3\ln 2 + (-2n+2)\ln 3$

$= 4\ln 2 + 3n\ln 2 + (-2n+2)\ln 3$

$= (3n+4)\ln 2 + (-2n+2)\ln 3$

$\therefore A = 3,\ B = 4,\ C = -2$ and $D = 2$

(b)

(ii) $S_\infty = \dfrac{128}{1-\dfrac{8}{9}}$

$= 1152$

$= 9(128)$

$= 9(\text{Length of the first plank})$

Since the total length of the planks $= 9$ times the length of the first plank, $\therefore$ the total length can never be greater 9 times the length of the first plank. (Shown)

$\therefore$ integer $k = 9$.

(iii) Given the total length of the planks cut will not exceed 1.05 m

i.e. $S_n \le 1050$

$$\frac{128\left[1-\left(\frac{8}{9}\right)^n\right]}{1-\frac{8}{9}} \le 1050 \quad \triangleleft \text{ use sum of GP: } \frac{a(1-r^n)}{1-r}, \text{ where } a = 128 \text{ and } r = \frac{8}{9}$$

$$1-\left(\frac{8}{9}\right)^n \ge \frac{1050}{1152}$$

$$\left(\frac{8}{9}\right)^n \ge \frac{17}{192}$$

$$n \le \frac{\ln\left(\frac{17}{192}\right)}{\ln\left(\frac{8}{9}\right)}$$

$$n \le 20.583$$

He needs to cut 20 pieces of the planks.

35
Solution

(a) Given that the 25th bar has length 5 m

i.e. $ar^{24} = 5$ ◁ use nth term of GP: ar^{n-1}, where $a = 128$ and $r = \frac{8}{9}$

$20r^{24} = 5$

$r = \sqrt[12]{\frac{1}{2}}$

Total length of all bars

$S_\infty = \dfrac{20}{1-\sqrt[12]{\frac{1}{2}}}$ ◁ use sum to infinity of GP: $S_\infty = \dfrac{a}{1-r}$, where $a = 20$ and $r = \sqrt[12]{\frac{1}{2}}$

$= 356.343$

< 357 (shown)

Since the total length of all the bars is 356.3 m, it can never be greater than 357 m. (Shown)

(b) Total length of all the bars of railway tack Y

$L = \dfrac{20\left[1-\left(\sqrt[12]{\frac{1}{2}}\right)^{25}\right]}{1-\sqrt[12]{\frac{1}{2}}} = 272.2573$ m ◁ use sum of GP: $\dfrac{a(1-r^n)}{1-r}$, where $a = 20$ and $r = \sqrt[12]{\frac{1}{2}}$ ‘

$\therefore$ the total length, L m, of all the bars of railway track Y is 272 m.

Length of 13th bar

$= 20\left(\sqrt[12]{\frac{1}{2}}\right)^{12}$ ◁ use nth term of GP: ar^{n-1}, where $a = 20$, $n = 12$ and $r = \sqrt[12]{\frac{1}{2}}$

$= 10$

The length of the13th bar is 10 m.

(c) Let the length of first bar of railway track Y be a_1 and the common difference be d

Given that the total length of the 25 bars is 272.2573 m,

i.e. $\frac{25}{2}\left[2a_1+(25-1)d\right]=272.2573$ ⊲ use sum of AP: $\frac{n}{2}\left[2a+(n-1)d\right]$, where $n=25$

$$\frac{25}{2}\left[2a_1+24d\right]=272.2573$$

$$a_1+12d=\frac{272.2573}{25} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Given that the length of the 25th bar is 5 m,

i.e. $a_1+24d=5$ ⊲ use nth term of AP: $a+(n-1)d$

$$a_1=5-24d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Substitute (2) into (1).

$$(5-24d)+12d=\frac{272.2573}{25}$$

$$d=5-\frac{272.2573}{25}$$

$$=-0.49086$$

$\therefore$ $=-0.491$ (3 s.f.)

The value of d is -0.491

Substitute $d=-0.49086$ into (2).

$$a_1=5-24(-0.49086)$$

$$=16.8 \text{ m} \quad \text{(3s.f.)}$$

The length of longest bar is 16.8 m

36
Solution

(a) Team A (follows Arithmetic Progression)

Team A covered 400 m on the first day and covered 5 m less than the vertical distance covered in the previous day.

Let the first day be the first term, $a = 400$. Distance covers on the second day = 395.

$\therefore$ the common difference, $d = -5$.

Let the number of days required for Team A to reach the peak be n.

Given that the total distance to cover is 8500, i.e. $S_n = 8500$.

$$\frac{n}{2}\left[2(400)+(n-1)(-5)\right] = 8500 \quad \triangleleft \text{ use sum of AP: } \frac{n}{2}\left[2a+(n-1)d\right], \text{ where } a = 400 \text{ and } d = -5$$

$$5n^2 - 805n + 17000 = 0$$

Using GC, $n = 25$ or $n = 136$ (rejected as Team A has already reached peak of the mountain when $n = 25$)

(b) Team B (follows Geometric Progression)

Team A covered 800 m on the first day and covered 90% of the distance covered in the previous day.

Let the first day be the first term = 800. Distance covers on the second day $= \frac{90}{100}(800)$.

$\therefore$ the common ratio = 0.9.

Let N be the day that Team A overtakes Team B.

Total distance covered by Team B in N days.

$$= \frac{800(1-0.9^N)}{1-0.9}$$

Total distance covered by Team A in N days.

$$= \frac{N}{2}\left[2(400)+(N-1)(-5)\right]$$

Given that Team A overtakes Team B in day N,

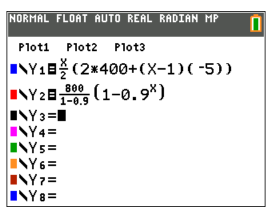
i.e. $$\frac{N}{2}\left[2(400)+(N-1)(-5)\right] > \frac{800(1-0.9^N)}{1-0.9}$$

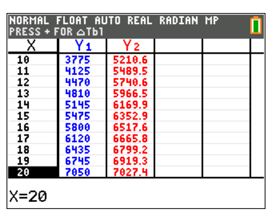
$$805N - 5N^2 > 16000(1-0.9^N)$$

Using GC,

when $N = 20$, $805N - 5N^2 > 16000(1-0.9^N)$

A will overtake B on the 20th day.





(c) Use sum to infinity formulae to find total distance covered by Team B

$$S_\infty = \frac{800}{1-0.9}$$
$$= 8000$$

Since ($S_\infty < 8500$), Team B will never be able to reach the peak.

(d) Total distance covered by Team B for the first 15 days

$$= \frac{800(1-0.9^{15})}{1-0.9}$$

$$= 6352.871$$

Remaining distance need to cover to reach the peak of the mountain

$$= 8500 - 6352.871$$

$$= 2147.129$$

Distance covered by Team B on the 15th day

$= 800(0.9^{15-1})$ ⊲ use nth term of GP: ar^{n-1}, where $a = 800$, $n = 15$ and $r = 0.9$

$= 183.014$

Distance covered by Team B on the 16th day

$= 183.014 \times 0.95$

$= 173.864$

Consider the 16th day be first term for the remaining distance need to cover to reach the peak of the mountain

$\therefore$ the first term of new GP is 173.864, the ratio of the new GP $= 0.95$.

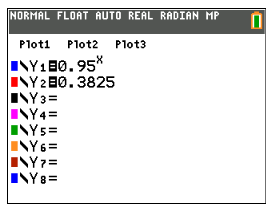
Total remaining distance need to cover $= 2147.129$

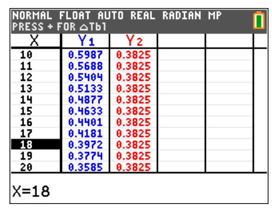
$$\frac{173.864(1-0.95^n)}{1-0.95} = 2147.129$$

$$0.95^n = 0.38253$$

Using GC, $n = 18.7$

$\therefore$ Team B will take $15 + 19 = 34$ days





Team A takes 25 days while Team B takes 34 days. Hence, Team A will reach the peak first.

37
Solution

(a) Swimmer A (follows Arithmetic Progression)

Let the time taken to complete the first lap be the first term $= T$

Each subsequent lap takes 1.5 seconds longer than the previous lap. $\therefore$ common difference $= 1.5$

Total time taken to complete the distance of 4 km (i.e. 80 laps)

$$= \frac{80}{2}\left[2T + (80-1)1.5\right]$$
$$= 80T + 4740$$

For Swimmer A to complete within the required time interval,

i.e. $2\frac{1}{3}$ hours $\le 80T + 4740 \le 2\frac{5}{6}$ hours ◁ convert hours to seconds

$$\left(2\frac{1}{3}\right)(60)(60) \le 80T + 4740 \le \left(2\frac{5}{6}\right)(60)(60)$$
$$8400 \le 80T + 4740 \le 10200$$
$$45.75 \le T \le 68.25$$

Set of values of T is $\{T \in \mathbb{R} : 45.75 \le T \le 68.25\}$

(b) Swimmer B (follows Geometric Progression)

Let the time taken to complete the first lap be the first term $= t$

Each subsequent lap takes 1.5% more than the time for the previous lap. $\therefore$ common ratio $= (1+1.5\%) = 1.015$

Total time taken to complete the distance of 4 km (i.e. 80 laps)

$$= \frac{t(1.015^{80}-1)}{1.015-1}$$
$$= \frac{200t}{3}(1.015^{80}-1)$$

For Swimmer B to complete within the required time interval,

i.e. $2\frac{1}{3}$ hours $\le \frac{200t}{3}(1.015^{80}-1) \le 2\frac{5}{6}$ hours ◁ convert hours to seconds

$$8400 \le \frac{200t}{3}(1.015^{80}-1) \le 10200$$
$$55.0059 \le t \le 66.7928$$
$$55.01 \le t \le 66.79 \quad \text{(correct to 2 dp)}$$

Set of values of t is $\{t \in \mathbb{R} : 55.01 \le t \le 66.79\}$

(c) To completing a distance of 1.8 km means the swimmer requires to swim 36 laps.

For swimmer A :

Total time taken to complete 36 laps in 50 minutes

i.e. $\frac{36}{2}\left[2T+(36-1)1.5\right]=50 \text{ mins}$ ⊲ convert minutes to seconds

$$\frac{36}{2}\left[2T+(36-1)1.5\right]=50(60) \text{ seconds}$$

$$T=57.083333 \text{ seconds}$$

$\therefore$ time taken for swimmer A to complete first lap is 57.083333 seconds.

Time taken for swimmer A to swim 80th lap

$=57.083333+(80-1)(1.5)$ ⊲ use nth term of AP: $a+(n-1)d$, where $a=57.083333$, $n=80$ and $d=1.5$

$=175.58333$

For swimmer B :

Total time taken to complete 36 laps in 50 minutes

i.e. $\frac{t(1.015^{36}-1)}{1.015-1}=50 \text{ mins}$ ⊲ convert minutes to seconds

$$\frac{t(1.015^{36}-1)}{1.015-1}=(50)(60) \text{ seconds}$$

$$t=63.457186 \text{ seconds}$$

$\therefore$ time taken for swimmer A to complete first lap is 63.457186 seconds.

Time taken for swimmer A to swim 80th lap

$=63.457186\,(1.015)^{80-1}$ ⊲ use nth term of GP: ar^{n-1}, where $a=63.457186$, $n=80$ and $r=1.015$

$=205.73024$

Swimmer A is faster in his 80th lap.

38
Solution

(a) Estate A (follows Arithmetic Progression)

Let the time that the company take to furnish the first flat be the first term, $a = 160$ and the total time that the company take to furnish first n houses be S_n.

Each successive flat the company furnished took m hours less than the previous one, i.e. common difference = m.

Given that the company took a total of 3800 hours to furnish the first 25 flats,

i.e. $S_{25} = 3800$

$$\frac{25}{2}\left[2(160) - m(25-1)\right] = 3800$$

$$320 - 24m = 304$$

$$m = \frac{2}{3}$$

Time to furnish next 25 houses

$S_{50} - S_{25}$

$$= \frac{50}{2}\left[2(160) - \frac{2}{3}(50-1)\right] - 3800$$

$$= 3383\frac{1}{3}$$

The company takes $3383\frac{1}{3}$ hours to furnish the next 25 flats.

Alternative Method

Time needed to furnish the 26th house

$$u_{26} = 160 - \frac{2}{3}(26-1)$$

$$= 143\frac{1}{3}\text{ h}$$

Total time needed to furnish the next 25 houses

$$= \frac{25}{2}\left[2\left(143\frac{1}{3}\right) - \frac{2}{3}(25-1)\right]$$

$$= 3383\frac{1}{3}\text{ hours}$$

(b) Let u_n be the nth flat that the company has fully furnished.

When the company takes less than 140 hours to furnish nth flat,

i.e. $u_n < 140$ ⊲ use nth term of AP: $a + (n-1)d$, where $a = 160$ and $d = -\frac{2}{3}$

$$160 - \frac{2}{3}(n-1) < 140$$

$$n - 1 > 30$$

$$n > 31$$

$\therefore\ n \geq 32$

On 32th flat, the company frst takes less than 140 hours to fully furnish a flat.

(c) Estate B (follows Geometric Progression)

Let the time that the company to furnish the first flat be the first term $= 160$ and the total time that the company to furnish first N houses be S'_N.

Each successive flat the company furnished took 4% less time to furnish than the previous one, i.e. common ratio = 0.96.

Given that he company took a total of 3800 hours to furnish the first N flats,

i.e. $S'_N = 3800$

$$\frac{160(1-0.96^N)}{1-0.96} = 3800$$

$$1-0.96^N = 0.95$$

$$0.96^N = 0.05$$

$$N = \frac{\ln 0.05}{\ln 0.96}$$

$$= 73.4$$

There are 73 flats can be fully fumished in 3800 hours.

(d) Total time required to furnish flats in Estate B

$$S_\infty = \frac{160}{1-0.96} \quad \triangleleft \text{ use sum to infinity}$$

$$= 4000$$

$\therefore$ the total time taken will not exceed 4000 hours.

The geometric series is convergent. Therefore the time taken for furnishing a house will eventually be negligible in the long run. This is not possible in reality because there is a need of substantial amount of time required to furnish a flat.

(e) Let x be the number of man employed for Estate A. $\therefore$ the number of man employed for Estate B is $2x$.

$$\text{Productivity for } A = \frac{\text{flats fully furnished}}{\text{man-hours}}$$

$$= \frac{25}{3800x}$$

$$\text{Productivity for } B = \frac{\text{houses fully furnished}}{\text{man - hours}}$$

$$= \frac{73}{3800 \times 2x}$$

$$= \frac{73}{7600x}$$

Improvement in productivity

$$= \frac{\frac{73}{7600x} - \frac{25}{3800x}}{\frac{25}{3800x}}$$

$$= 46\%$$

The percentage change in productivity level is 46%.

39
Solution

(a) Plan A (follows Arithmetic Progression)

Consider the first GB of data that costs \$0.20 as the first term = 0.2

Given that each subsequent GB of data costs \$0.013 more than the previous GB, i.e. common difference = 0.013

(i) Cost of the nth GB of data

$= 0.2 + (n-1)(0.013)$ ⊲ use nth term of AP: $a + (n-1)d$, where a = 0.2 and d = –0.013

$= 0.187 + 0.013n$

The cost of the nth GB of data is $0.187 + 0.013n$

(ii) Maximum cost per GB under Plan A

$= 0.187 + 0.013(40)$

$= \$0.707$

The maximum cost per GB under Plan A is \$0.707.

(iii) Total cost for first 40GB $= \dfrac{40}{2}[2(0.2) + (40-1)(0.013)]$

$= \$18.14$

Number of additional GB (after 40th GB) $= \dfrac{30 - 18.14}{0.707}$

$= 16.775$ (5 s.f.)

Alfred should set the data limit of $40 + 16 = 56$ GB.

(b) Plan B (follows Geometric Progression)

Consider the first GB of data that costs \$1.30 as the first term = 1.30

Given that each subsequent GB of data costs x% of the previous GB, i.e. common ratio = $0.01x$

(i) The amount charged to the customer is a sum of a geometric progression with general term $= 1.3\left(\dfrac{x}{100}\right)^{n-1}$.

Since $\left|\dfrac{x}{100}\right| < 1$, the sum is convergent.

Thus, there is a limit to the amount charged to the consumer.

(ii)

$$\frac{1.3}{1-\frac{x}{100}} = 88$$

$$1-\frac{x}{100} = 0.014773$$

$$\frac{x}{100} = 0.98523 \text{ (5 s.f.)}$$

$$x = 98.523$$

$$= 98.5 \text{ (correct to 3s.f.)}$$

The value of x is 98.5.

40
Solution

(a) Consider Sophia odd-numbered day of training

$12, \ 14, \ 16, \ 18, \ ....$

The above sequence follows arithmetic progression.

Let first term, $a = 12$, common difference $d = 2$ and $n =$ number of days.

Let u_n be the nth day to complete 42 km.

$$\therefore \quad u_n = 42$$
$$12 + (n-1)(2) = 42$$
$$\therefore \quad n = 16$$

Hence it requires 16 odd numbered days to complete 42 km.

Starting on 1st day, the last day is given by $1 + (16-1)(2) = 31$.

Therefore Sophia completed 42 km on the 31st day of training. (shown)

(b) Distance that Arthur covered on the 1st day of training $= 12$ km

Distance that Arthur covered on the 2nd day of training $= 12 \times \left(1 + \frac{k}{100}\right) = 12\left(1 + \frac{k}{100}\right)$ km

Distance that Arthur covered on the 3rd day of training $= \left(12 \times \left(1 + \frac{k}{100}\right)\right)\left(1 + \frac{k}{100}\right) = 12\left(1 + \frac{k}{100}\right)^2$ km

The above sequence follows geometric progression.

$\therefore$ first term, $a = 12$, common ratio $r = \left(1 + \frac{k}{100}\right)$.

Given tht he will be running 42 km on the same day as Sophia, i.e. on the 31st day. $(n = 31)$

$$12\left(1 + \frac{k}{100}\right)^{31-1} = 42 \qquad \triangleleft \text{ using } u_n = ar^{n-1}$$

Using GC, $k = 4.26$ (2 d.p.)

The value of k is 4.26.

(c) Total distance covered by Sophia

$= 2 \times$ distance covered during 1 st 15 odd numbered days $+$ distance covered on 31st day

$= \left(2\left[\frac{15}{2}(2(12) + (15-1) \times 2)\right] + 42\right)$

$= 822$

Total distance covered by Arthur

$= \dfrac{12(1 - (1 + 0.42643)^{31})}{1 - (1 + 0.42643)}$

$= 745.516$

Total distance covered by both runners

$= 822 + 745.516$

$= 1567.516$

$= 1570$ (3s.f.)

Exercise 5

G Applications II (Intermediate)

41
Solution

(a) Amount at the end of June 2010

$= 150000\left(1+\frac{0.2}{100}\right)^6$

$= \$151809.02$

(b) Amount at the end of January 2010

$= (150000 - 1000)(1.002)$

$= \$149298$

(c) Taking Jan 2010 as the first month

Month	Beginning of month after withdrawal	Amount at end of month
1	$(150000-1000)$	$(150000-1000)(1.002)$
2	$(150000-1000)(1.002)-1000$	$[(150000-1000)(1.002)-1000](1.002)$ $=150000(1.002)^2-1000(1.002^2+1.002)$
3	$150000(1.002)^2-1000(1.002^2+1.002)$ -1000	$[150000(1.002)^2-1000(1.002^2+1.002)-1000](1.002)$ $=150000(1.002)^3-1000((1.002^3+1.002^2+1.002)$
...	...	...

Observe the sequence in the table.

Amount at the end of the n-th month

$= 150000(1.002)^n - 1000[(1.002)^n + ... + (1.002)^2 + 1.002]$

$= 150000(1.002)^n - 1000\left[\frac{1.002(1.002^n - 1)}{1.002 - 1}\right]$

$= 150000(1.002)^n - 501000(1.002^n - 1)$

$= 501000 - 351000(1.002)^n$, where $k = 501000$ (Shown)

(d) $501000 - 351000(1.002)^n \le 0$

$$(1.002)^n \ge \frac{501000}{351000}$$

$$n \ge \frac{\ln\left(\frac{501}{351}\right)}{\ln(1.002)}$$

$$n \ge 178.09$$

Therefore the account is depleted in 179th month which is November 2024.

(e) Amount for last withdrawal

$= 501000 - 351000(1.002)^{178}$

$= \$87.87$

42
Solution

(a)

n	Beginning of the year	End of the year
1	50000	50000(1.08)
2	$50000(1.08)+x$	$50000(1.08)^2+1.08x$
3	$50000(1.08)^2+1.08x+x$	$50000(1.08)^3+(1.08)^2x+1.08x$

Observe the sequence in the table.

Amount at the end of nth year

$$= 50000(1.08)^n + 1.08^{n-1}x + 1.08^{n-2}x + \ldots + 1.08^2x + 1.08x$$

$$= 50000(1.08)^n + x\left[1.08^1 + 1.08^2 + \ldots + 1.08^{n-2} + 1.08^{n-1}\right]$$

$$= 50000(1.08)^n + \frac{1.08x(1.08^{n-1}-1)}{1.08-1}$$

$$= 50000(1.08)^n + 13.5x(1.08^{n-1}-1) \quad \text{(Shown)}$$

(b) Interest earned at the end of the 10 th year

= (Amount in the account including interest) – (Amount deposited in the account for 10 years)

$= 50000(1.08)^{10} + 13.5(5000)(1.08^{10-1}-1) - \left[50000 + 9(5000)\right]$ ◁ take $x = 5000$ and $n = 10$

$= 175379 - 95000$

$= 80579$

The interest earned at the end of the 10 th year is \$80 379.

(c) Given that the interest rate is $r\%$, the first deposit is \$50000 and the yearly fixed deposit is \$5000

$\therefore$ the amount of money in the savings account with interest rate after 10 years

is $$50000\left(1+\frac{r}{100}\right)^{10} + \frac{R(5000)(\left(1+\frac{r}{100}\right)^{10-1}-1)}{\left(1+\frac{r}{100}\right)-1}$$

Learning point:

Substitute $n = 10$, $x = 5000$ and replace 1.08 by $\left(1+\frac{r}{100}\right)$ into $50000(1.08)^n + \frac{1.08x(1.08^{n-1}-1)}{1.08-1}$

to obtain $50000\left(1+\frac{r}{100}\right)^{10} + \frac{R(5000)(\left(1+\frac{r}{100}\right)^{10-1}-1)}{\left(1+\frac{r}{100}\right)-1}$.

For Mr Tan needs to have \$300000 in his account after 10 years,

$$\therefore\ 50000\left(1+\frac{r}{100}\right)^{10} + \frac{R(5000)(\left(1+\frac{r}{100}\right)^{10-1}-1)}{\left(1+\frac{r}{100}\right)-1} = 300000$$

Using GC, $r = 15.1$

The value of r that will allow Mr Tan to have \$300000 in his account after 10 years is 15.1.

43
Solution

(a)(i) Amount in account at the end of 1st year

$= \frac{5}{100}y + y$

$= 1.05y$

Mary has $\$1.05y$ in the investment plan at the end of 1 year.

(ii)

Year	Total amount in the investment plan at the end of the year
1	$1.05y$
2	$[1.05y + y] \times 1.05$ $= 1.05^2 y + 1.05y$
3	$[1.05^2 y + 1.05y + y] \times 1.05$ $= 1.05^3 y + 1.05^2 y + 1.05y$
...	...

Total Amount in the investment plan at the end of nth year

$= 1.05^n y + 1.05^{n-1} y + \cdots + 1.05y$

$= 1.05^1 y + 1.05^2 y + ... + 1.05^n y + 1.05^{n-1} y$ ⊲ the series is GP with first term $1.05y$ and common ratio 1.05

$= \frac{1.05y(1.05^n - 1)}{1.05 - 1}$

$= 21y(1.05^n - 1)$ (Shown)

(iii) For Mary to have at least $\$15y$ in her plan,

i.e. $21y(1.05^n - 1) \geq 15y$

$21(1.05^n - 1) \geq 15$

$1.05^n \geq \frac{12}{7}$

$n \geq 11.05$ (correct to 2dp)

After 12 complete years, Mary would have at least $\$15y$ in her plan.

(b) At the end of n years, the interest that Tom withdrawal

$0.05(3x) + 0.05(5x) + 0.05(7x) + ... + n\text{th term}$

$= 0.05x[3 + 5 + ... + n\text{th term}]$ ⊲ the series $3 + 5 + ... + n$th term shows an AP series

$= 0.05x\left[\frac{n}{2}(2(3) + (n-1)(2))\right]$ ⊲ use $u_n = a + (n-1)d$, where $a = 3$ and $d = 2$

$= 0.05x(2n + n^2)$

The total amount of interest Tom has withdrawn after n years is $\$0.05x(2n + n^2)$.

44
Solution

(a) Given that $I = 3$, $\therefore$ annual compounded interest rate is 3%.

Year	Start	End
1	1200	$1200(1.03)$
2	$1200(1.03)+1000$	$1200(1.03)^2+1000(1.03)=2303.08$

Albert's account balance on the last day of 2022 is \$2303.08 (Shown)

(b)(i)

n	Amount on the first day of the nth year (\$)	Amount on the last day of nth year (\$)
1	1200	$1200r$ where $r=\dfrac{100+x}{100}$
2	$1200r+1000$	$1200r^2+1000r$
3	$1200r^2+1000r+1000$	$1200r^3+1200r^2+1000r$
	...	...

Observe the sequence in the table.
Amount at the end of 15th year (i.e. for the year 2035)
$=1200r^{15}+1000r^{14}+...+1000r$ $\triangleleft$ substitute $r=15$

Given that the account balance on the last day of the year 2035 is \$19000,

i.e. $$1200r^{15}+1000r^{14}+...+1000r=19000$$
$$1200r^{15}+1000(r+r^2+...+r^{14})=19000$$
$$1200r^{15}+1000\left[\frac{r(r^{14}-1)}{r-1}\right]=19000$$
$$12r^{15}+10\frac{r(r^{14}-1)}{r-1}-190=0$$

From GC, $r=1.0271$

$\therefore \dfrac{100+x}{100}=1.0271$

From GC, $x=2.7$ (correct to 1 d.p.)

(ii) For Albert to accumulate more than \$500000 in his bank account,

then $1200r^n+1000\dfrac{r(r^{n-1}-1)}{r-1}>50000$

Taking $r=1.027$ (from **(b)(i)**)

$\therefore 1200(1.027)^n+1000\dfrac{1.027(1.027^{n-1}-1)}{1.027-1}>50000$

n	$1200r^n+1000\dfrac{r(r^{n-1}-1)}{r-1}$
31	$49294<50000$
32	$51652>50000$
33	$54073>50000$

Refer to the table.
When $n=32$, Albert accumulates more than \$51652

$\therefore$ the minimum number of years required for Albert to accumulate more than \$50000 in his bank account is 32.

\

(c)

n	Amount at start of the nth year (\$)	Amount at the end of the nth year (\$)	Amount of bonus earned at the end of nth year (\$)
1	1000	$1000+2a$	$2a$
2	$1000+2a+1000=2(1000)+2a$	$2(1000)+2a+4a$	$4a$
3	$2(1000)+2a+4a+1000$	$3(1000)+2a+4a+6a$	$6a$
	...	...	...
	...	...	...
n		$n(1000)+(2+4+6+...+2n)a$	$2na$

Refer to the above table (last column)

Toral amount of bonuses earned

$=2a+4a+6a+....+2na$

$=(2+4+6+...+2n)a$ $\triangleleft$ apply sum of AP formulae, where first term = $2a$, last term = $2n$ and number of terms = n

$=\frac{n}{2}a(2+2n)$

$=an(1+n)$

The total amount of bonuses given to Ling at the end of n years is \$ $an(1+n)$.

(d) Total amount of money in Ling's investment account at the end of the nth year

$=n(1000)+a(2+4+6+...+2n)$ $\triangleleft$ refer to the table in **(c)**

$=n(1000)+\frac{n}{2}a(2+2n)$

When $n=20$,

$=20(1000)+\frac{20}{2}a(2+2\times 20)$ .... (1)

Total amount of money in Ling's investment account at the end of the nth year

$=1200r^n+1000\frac{r(r^{n-1}-1)}{r-1}$

When $n=20$ and $r=1.02$

$=1200(1.02)^{20}+1000\frac{(1.02)((1.02)^{19}-1)}{(1.02)-1}$ .... (2)

Given that the amount in Ming's bank account equals the amount in Ling's investment account at the end of the 20th year

Equating (1) and (2)

$$1200(1.02)^{20}+1000\frac{(1.02)((1.02)^{19}-1)}{(1.02)-1}=20(1000)+\frac{20}{2}a(2+2\times 20)$$

Using GC, $a=12.10$ (correct to 2 d.p. since a is amount of money)

$\therefore\ a=\$12.10$

For this value of a, Ling investment plan generates a higher as Ming contributed an extra \$200 in total although both Ming and Ling cash out the same amount at the end of the 20th year.

45
Solution

(a) Amount saved from January 31, 2001 on 31 December 2023, i.e. 50 months

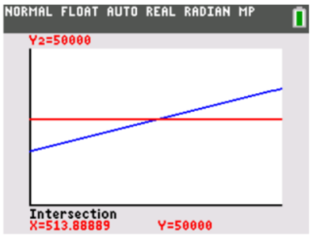
$= a + (a+50) + [a + 2(50)] + \ldots + [a + 35(50)]$

$= \dfrac{30}{2}[2a + (36-1)(50)]$

Given that she saves at least \$ 50000,

$\therefore \quad \dfrac{30}{2}[2a + (36-1)(50)] \geq 50000$

Using GC,



Thus, smallest value of $a = 513.89$

(b)

n	Month	Amount owing at the start of n month	Amount owing at the end of n month
1	Jan 2024	$400\,000(1.01)$	$400\,000(1.001) - x$
2	Feb 2024	$400\,000(1.001)^2 - x(1.001)$	$400\,000(1.001)^2 - x(1.001)^2 - x$
3	Mar 2024	$400\,000(1.001)^3 - x(1.001)^2 - x(1.001)$	$400\,000(1.001)^3 - x(1.001)^2 - x(1.001) - x$
	$\ldots$	$\ldots$	$\ldots$
	$\ldots$	$\ldots$	$\ldots$
n		$400\,000(1.001)^n - x(1.001)^{n-1} - \ldots - x(1.001)$	$400\,000(1.001)^n - x(1.001)^{n-1} - \ldots - x(1.001) - x$

Refer to the table.

Amount owed at the end of n month

$= 400000(1.001)^n - x(1.001)^{n-1} - x(1.001)^{n-2} - \ldots - x$

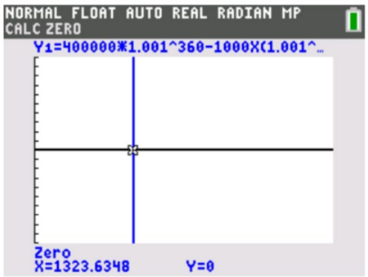
$= 400000(1.001)^n - \left[x(1.001) + x(1.001)^2 + \ldots + x(1.001)^{n-2} + x(1.001)^{n-1}\right]$

$= 400000(1.001)^n - \dfrac{x(1.001^n - 1)}{1.001 - x}$

$= 400000(1.001)^n - 1000x(1.001^n - 1) \quad$ (Shown)

(c)(i) Let A_n be the amount owed at the end of n months.

$\therefore\ A_n = 400000(1.001)^n - 1000x(1.001^n - 1)$ .... (1)



Given that the loan is repaid in 360 monthly repayments, i.e. $A_{360} \le 0$

$\therefore\ 400000(1.001)^{360} - 1000x(1.001^{360} - 1) \le 0$

Using GC, $x = 1323.6348 = 1323.63$ (correct to 2 dp)

Analytical method

$(1.001)^{360} 400\,000 - 1000x(1.001^{360} - 1) \le 0$

$$x \ge \frac{(1.001)^{360} 400\,000}{1000(1.001^{360} - 1)}$$

$$x \ge 1323.6348$$

Hence $x = 1323.64$ (correct to 2 d.p.)

(c)(ii) Total interest paid on the loan

$=$ (Total amount paid for 360 months including interest) $-$ (Amount of loan)

$= 1323.63 \times 360 - 400000$

$= 76508.528$

$= 76508.53$ (correct to 2 dp)

$\therefore$ Total interest paid on the loan is \$76508.53.

(d)(i) Amount owed at the end of k months if Kim repays the loan at the rate of \$1600 per month

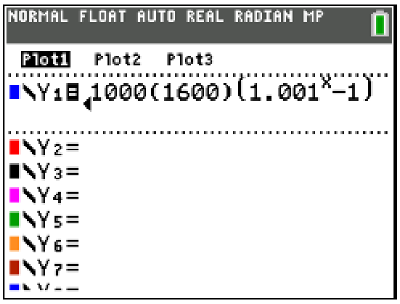
Substitute $n = k$ and $x = 1600$ into (1)

$A_n = 400000(1.001)^k - 1000(1600)(1.001^k - 1) < 1600$

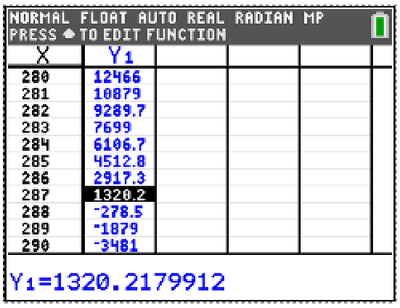
Using GC, $k = 287$

$\therefore$ Kim repays the loan at 1600 per month for $k = 287$ months

Amount owed at the end of 287 months is \$1320.22 (refer to GC)



The final repayment of $y = 1320.21799 \times 1.001 = 1321.51$



(d)(ii) Total interest paid in **(d)(i)**

$= 1600 \times 287 + 1321.54 - 400000$

$= 60521.54$

Total saving compared with **(c)(i)**

$= 76508.53 - 60521.54$

$= 15986.99$

$= 15990$ (correct to 4 significant figures)

46
Solution

(a)

n	Amount at the beginning of nth month	Amount at the end of nth month
1	40000	40000(1.005)
2	40000(1.005)	$40000(1.005)^2$
3	$40000(1.005)^2 - x$	$40000(1.005)^3 - 1.005x$
4	$40000(1.005)^3 - 1.005x - x$	$40000(1.005)^3 - (1.005)^2 - 1.005x$
5	$40000(1.005)^4 - 1.005^2 x - 1.005x - x$	

Amount at the beginning of 5th month

$= 40000(1.005)^4 - 1.005^2 x - 1.005x - x$

$$= 40000(1.005)^4 - \frac{x(1.005^3 - 1)}{1.005 - 1}$$

$= 40000(1.005)^4 - 200x(1.005^3 - 1)$ (Shown)

(b) Amount at the beginning of nth month

$= 40000(1.005)^{n-1} - .... - 1.005^2 x - 1.005^1 x - 1.005^0 x$

Amount at the beginning of 60th month

$= 40000(1.005)^{59} - .... - 1.005^2 x - 1.005^1 x - 1.005^0 x$

$= 40000(1.005)^{59} - 200x(1.005^{58} - 1)$

Douglas wishes to repay his in 5 years, $n = 60$ months

$\therefore \quad 40000(1.005)^{59} - 200x(1.005^{58} - 1) \le 0$

$$200x(1.005^{58} - 1) \ge 40000(1.005)^{59}$$

$$x \ge \frac{40000(1.005)^{59}}{200(1.005^{58} - 1)}$$

$$x \ge 800.17$$

The minimum repayment amount is \$800.17

(c) Amount interest bank earned

$= \$(800.17(58) - 40000)$

$= \$6410.06$

$= \$6410$ (nearest dollar)

47
Solution

(a) $1.01(20000-x)=20000$

$$1.01x=0.01(20000)$$

$$x=198.02$$

(b)

Number of payments	Amount owed after each payment in the middle of the month
1	$20000-x$
2	$1.01(20000-x)-x$ $=1.01(20000)-1.01x-x$
3	$1.01[1.01(20000)-1.01x-x]-x$ $=1.01^2(20000)-1.01^2x-1.01x-x$
...	...

Amount owed at the end of n month

$$=1.01^{n-1}(20000)-1.01^{n-1}-1.01^{n-2}x+...+1.01x+1.01^0x$$

$$=1.01^{n-1}(20000)-x(+1.01^0+1.01+...+1.01^{n-2}+1.01^{n-1})$$ ⊲ use sum of GP formulae

$$=1.01^{n-1}(20000)-x\frac{1.01^n-1}{1.01-1}$$

For the loan to be paid in full after the nth payment,

let $1.01^{n-1}(20\,000)-k\dfrac{1.01^n-1}{1.01-1}=0$

$$1.01^{n-1}(20\,000)=k\frac{1.01^n-1}{0.01}$$

$$k=\frac{200(1.01^{n-1})}{1.01^n-1}$$ (Shown) .... (1)

Alternative Method

Number of payments	Amount owed at the end of each month
1	$1.01(20000-x)$
2	$1.01[1.01(20000-x)-x]$ $=1.01^2(20000)-1.01^2x-1.01x$
...	...

Amount of owed at the end of $(n-1)$ month after number of $(n-1)$ payment

$$=1.01^{n-1}(20000)-1.01^{n-1}(20000)-....-1.01^2(20000)-1.01^1(20000)$$

$$=1.01^{n-1}(20000)-\left[1.01^1(20000)+1.01^2(20000)+....+1.01^{n-1}(20000)\right]$$

$$1.01^{n-1}(20\,000)-x\left[\frac{1.01(1.01^{n-1}-1)}{1.01-1}\right]$$

For the loan to be paid in full after the nth payment, then

$$1.01^{n-1}(20\,000)-k\left[\frac{1.01(1.01^{n-1}-1)}{1.01-1}\right]-k=0$$

$$1.01^{n-1}(20\,000)=k\left[\frac{1.01(1.01^{n-1}-1)}{0.01}\right]+k$$

$$k\left[\frac{1.01(1.01^{n-1}-1)+0.01}{0.01}\right]=1.01^{n-1}(20\,000)$$

$$k\left[\frac{1.01^{n-1}-1}{0.01}\right]=1.01^{n-1}(20\,000)$$

$$k=\frac{200(1.01^{n-1})}{1.01^{n}-1} \quad \text{(shown)}$$

(c) For the loan to be fully paid in 3 years ($n=36$ months)

$$k=\frac{200(1.01^{36-1})}{1.01^{36}-1} \quad \triangleleft \text{ substitute } n=36 \text{ into } k=\frac{200(1.01^{n-1})}{1.01^{n}-1}$$

$k=657.709$

$k=657.71$ (correct to nearest cents)

Hence, for Thomas to fully pay up the loan in exactly 3 years, he should be paying a monthly amount of \$657.71

48
Solution

Month	Number of nth payment	Amount owed at the start of nth month	Amount owed at the end of nth month
Jan 2018	0	P	$P(1.01)$
Feb 2018	1	$\left(P(1.01)-800\right)$	$\left(P(1.01)-800\right)1.01$ $=P(1.01)^2-800(1.01)$
Mar 2018	2	$P(1.01)^2-800(1.01)-800$	$\left[P(1.01)^2-800(1.01)-800\right](1.01)$ $=P(1.01)^3-800(1.01)^2-800(1.01)$
Apr 2018	3	$P(1.01)^3-800(1.01)^2-800(1.01)-800$	$\left[P(1.01)^3-800(1.01)^2-800(1.01)-800\right](1.01)$ $=P(1.01)^4-800(1.01)^3-800(1.01)^2-800(1.01)$

(a) Outstanding amount at the end of the month after nth repayment

$=P(1.01)^{n+1}-800(1.01)^n-.....-800(1.01)^2-800(1.01)^1$

$=P(1.01)^{n+1}-800(1.01^n+1.01^{n-1}+...+1.01)$

$=P(1.01)^{n+1}-800\dfrac{(1.01)(1.01^n-1)}{1.01-1}$

$=P(1.01)^{n+1}-80000(1.01^{n+1}-1.01)$

$=(P-80000)(1.01)^{n+1}+80800$ (Shown) .... (1)

(b)(i) Substitute $P=20\,000$ into (1)

Outstanding amount at the end of the month after n-th repayment

$=(20\,000-80000)(1.01)^{n+1}+80800$

$=80800-60000(1.01)^{n+1}$ .... (2)

For complete payment of the loan, the outstanding amount at the end of the month after n-th repayment ≥ 0

$\therefore$ $80800-60000(1.01)^{n+1}\geq 0$

$$(1.01)^{n+1}\leq\frac{80800}{60000}$$

$$n+1\leq\frac{\ln\frac{80800}{60000}}{\ln 1.01}$$

$$n+1\leq 29.9$$

$$n\leq 28.9$$

Total number of installment payments he made , including the final payment that he repaid the loan is 29.

(b)(ii) Substitute $n=28$ into (2)

$=80800-60000(1.01)^{28+1}$

$=$ 729.77

Outstanding amount at the end of the month after 28-th repayment is 729.77

$\therefore$ the last payment (i.e. 29-th repayment) is \$730 (correct to the nearest dollar)

(c) Given monthly installment = 800

Month	Salary	Saving for the month
1	2400	$0.2(2400-800)=0.2(1600)$
2	$2400+10$	$0.2(2400+10-800)=0.2(1600+10)$
3	$2400+2(10)$	$0.2\left[2400+2(10)-800\right]=0.2\left[1600+2(10)\right]$
4	$2400+3(10)$	$0.2\left[2400+3(10)-800\right]=0.2\left[1600+3(10)\right]$

Saving at the end of n month

$=0.2\left[1600+(n-1)(10)\right]$

Total savings at the end of n months from Feb 2018

$=0.2(1600)+0.2(1600+10)+...+0.2\left[1600+(n-1)(10)\right]$

$=0.2\left\{1600+(1600+10)+\left[1600+(n-1)(10)\right]\right\}$ $\triangleleft$ series is AP and use sum of AP: $\dfrac{n}{2}(\text{first term}+\text{last term})$

$=0.2\left\{\dfrac{n}{2}[1600+(1600+(n-1)(10)]\right\}$

$=0.2\left\{\dfrac{n}{2}([3200+10n-10)]\right\}$

$=0.1\left[n(3190+10n)\right]$

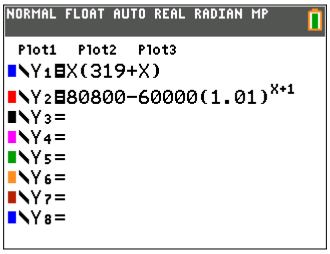
$=n(319+n)$

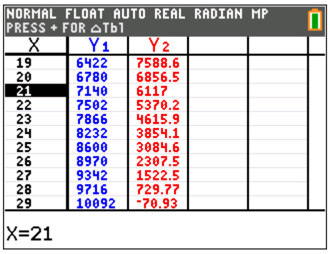
For him to clear his loan with his savings after his nth repayment,

Total savings at the end of n months from Feb 2018 $\geq$ Outstanding amount at the end of the month after n-th repayment

i.e. $\quad n(319+n)\geq 80\,800-60\,000(1.01)^{n+1}$

Using G.C





X	Y1	Y2
19	6422	7588.6
20	6780	6856.5
21	7140	6117
22	7502	5370.2
23	7866	4615.9
24	8232	3854.1
25	8600	3084.6
26	8970	2307.5
27	9342	1522.5
28	9716	729.77
29	10092	-70.93

From the table, $n=21$

Therefore he will clear his loan in Oct 2019.

49
Solution

(a) Total distance Lucas will cover in his third set of jump and slide

$= 2(0.95)^2 + (0.01 + 2(0.01))$
$= 1.835 \text{ m}$

Learning point:
Distance Lucas covers in his first set of jump and slide $= 2 + 0.01$
Distance Lucas covers in his second set of jump and slide $= 2(0.95) + 0.01 + 0.01$
Distance Lucas covers in his third set of jump and slide $= 2(0.95)(0.95) + 0.01 + 0.01 + 0.01$

(b) Let n be the number of sets of jumps and slides.

For the number of sets of jumps and slides Lucas needs to complete in order to cross the 15-metre mark,

$$\frac{2(1-0.95^n)}{1-0.95} + \frac{n}{2}(2(0.01) + (n-1)(0.01) \geq 15$$

Using GC, $n \geq 8.828$

The least number of sets of jumps and slides that Lucas needs to complete in order to cross the 15-metre mark is 9.

Let T_n be the single jump distance on nth week

$T_{10} = 2(1.02)^9 = 2.39 < 2.5$ or $T_{11} = 2(1.02)^{10} = 2.44 < 2.5$

Glen's belief is not valid

50
Solution

(a)(i) Let u_n m denote the distance run by the athelete in stage n.

$$\left.\begin{array}{l} \text{Stage } 1: u_1 = 8 \\ \text{Stage } 2: u_2 = 16 \\ \text{Stage } 3: u_3 = 24 \\ \text{Stage } 4: u_4 = 32 \\ \cdots \\ \cdots \end{array}\right\}$$

From the above, the sequence follows an Arithmetic Progression
First term $a = 8$ and common difference $d = 8$

Distance run by the athlete after completing the first 10 stages

$$= u_1 + u_2 + u_3 + ... + u_{10}$$
$$= \frac{10}{2}[2(8) + (10-1)(8)]$$
$$= 440 \text{ m}$$

(a)(ii) Distance run by the athlete after completing the first n stages

$$= \frac{n}{2}[2(8) + (n-1)(8)]$$
$$= 4n(1+n)$$

To complete at least 5 km,

$$4n(1+n) \geq 5000$$
$$4n^2 + 4n \geq 5000$$
$$\therefore \quad n^2 + n - 1250 \geq 0$$

$n \geq 34.859$ or $n \leq -35.859$ (rejected as $n > 0$)

The least number of stages that the athlete needs to complete to run at least 5 km is 35

(b) Let v_n m denote the distance run by the athelete in stage n.

$$\left.\begin{array}{l} \text{Stage } 1: v_1 = 8 \\ \text{Stage } 2: v_2 = 16 \\ \text{Stage } 3: v_3 = 32 \\ \text{Stage } 4: v_4 = 64 \\ \cdots \\ \cdots \end{array}\right\}$$

From the above, the sequence follows a Geometric Progression
First term $a = 8$ and common ratio $r = 2$

Distance run by the athlete after completing the first n stages

$= v_1 + v_2 + v_3 + \ldots + v_n$

$= \dfrac{8(2^n - 1)}{2-1}$ ◁ use sum of GP formulae with first term 8 and common ratio 2

$= 8(2^n - 1)$ m

Let $8(2^n - 1) = 10000$

$n = 10.3$

The athlete needs stage 10.3 to complete 10 km

Now consider stage 10.

Distance run by the athlete after completing 10 stages

$= 8(2^{10} - 1)$

$= 8184$ m

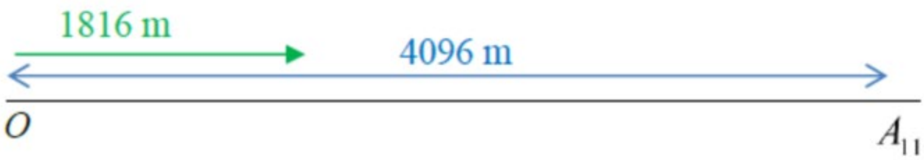
$\therefore$ After running 10 km, the athlete has completed 10 stages and $10000 - 8184 = 1816$ m of the stage 11.

Now consider stage 11.

Distance run by the athlete after completing 11 stages

$= 8(2^{11} - 1)$

$= 16376$ m



Note that the distance from O to A_{11}

$= \dfrac{1}{2}(8(2)^{11-1})$

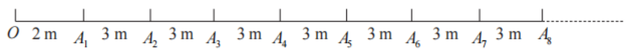
$= 4096$ m

He is 1816 m away from O and running towards A_{11}.

(c)(i) OA_n ◁ the sequence follows AP with first term 2, common difference 3 and number of term $(n-1)$

$= 2 + 3(n-1)$

$= 3n - 1$



A_1A_n

$= 3(n-1)$

$= 3n - 3$

(c)(ii) Total distance $OA_n + A_1A_n + A_2A_n + \ldots + A_{n-2}A_n + A_{n-1}A_n$

$= OA_n + A_1A_n + A_2A_n + \ldots + A_{n-2}A_n + A_{n-1}A_n$

$= (3n-1) + (3n-3) + (3n-6) + (3n-9) + \ldots. + 6 + 3$

$= (3n-1) + \dfrac{n-1}{2}(3n-3+3)$

$= (3n-1) + \dfrac{(n-1)(3n)}{2}$

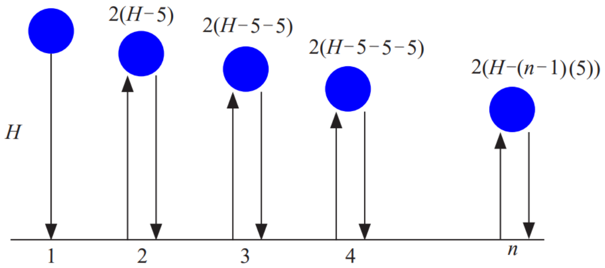
$= 3n - 1 + \dfrac{3}{2}n^2 - \dfrac{3}{2}n$

$= \dfrac{3}{2}n^2 + \dfrac{3}{2}n - 1$ (Shown)

Note

The series $(3n-3) + (3n-6) + (3n-9) + \ldots. + 6 + 3$ follows AP series

51
Solution



(a) Height reached after 1st bounce $= H - 5$

Height reached after 2nd bounce $= H - (2)5$

Height reached after 3rd bounce $= H - (3)5$

...

Height reached after the $(n-1)$th bounce $= H - (n-1)(5)$

(b) Total distance travelled by the ball just before it rebounds off the floor for the nth time

$$= H + 2(H-5) + 2(H-2(5)) + 2(H-3(5)) + ... + 2(H-(n-1)(5))$$

$$= H + \underbrace{2H + 2H + ... + 2H}_{(n-1)\text{terms}} - 2[5 + (5+5) + (5+5+5) + ... + (n-1)(5)]$$

$$= H + 2(n-1)H - 2\left[\frac{n-1}{2}(5 + (n-1)5)\right]$$

$$= H + 2(n-1)H - 5n(n-1)$$

Alternative Method

Total distance travelled by the ball just before it rebounds off the floor for the nth time

$$= H + 2(H-5) + 2(H-2(5)) + 2(H-3(5)) + ... + 2(H-(n-1)(5))$$

$$= H + \underbrace{2H + 2H + ... + 2H}_{(n-1)\text{terms}} - 2[5 + (5+5) + (5+5+5) + ... + (n-1)(5)]$$

$$= H + 2(n-1)H - 2\left\{\frac{n-1}{2}[2 \times 5 + ((n-1)-1)5]\right\}$$

$$= H + 2(n-1)H - (n-1)(10 + 5n - 10)$$

$$= H + 2(n-1)H - 5n(n-1)$$

(c) - The ball only travels vertically

- Upon hitting the floor, the tennis ball rebounds instantly

(d) Time taken between 1^{st} and 2^{nd} bounce $= 1.2$

Time taken between 2^{nd} and 3^{rd} bounce $= 1.2(0.75)$

Time taken between 3^{rd} and 4^{th} bounce $= 1.2(0.75)^2$

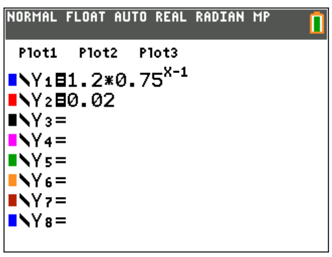
...

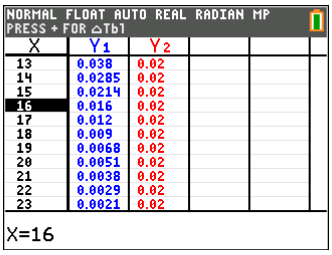
Time taken between m^{th} and $(m+1)^{\text{th}}$ bounce $= 1.2(0.75)^{m-1}$

For the ball to take less than 0.02 seconds between the mth and $(m+1)$th bounce

i.e. $1.2(0.75)^{m-1} < 0.02$

Using GC, The least $m = 16$





X	Y1	Y2
13	0.038	0.02
14	0.0285	0.02
15	0.0214	0.02
16	0.016	0.02
17	0.012	0.02
18	0.009	0.02
19	0.0068	0.02
20	0.0051	0.02
21	0.0038	0.02
22	0.0029	0.02
23	0.0021	0.02

Alernative Method

$$1.2(0.75)^{m-1} < 0.02$$

$$(0.75)^{m-1} < \frac{0.02}{1.2}$$

$$m-1 > \frac{\ln\left(\frac{0.02}{1.2}\right)}{\ln(0.75)}$$

$$m-1 > 14.232$$

$$m > 15.232$$

$\therefore$ the least integer m such that it takes less than 0.02 seconds between the mth and $(m+1)$th bounceLeast $m = 16$

(e) Total time taken before the tennis balls comes to a stop

$= \frac{1.2}{1-0.75} + 2$

$= 6.8$ s

Total time taken before the tennis ball comes to a stop is 6.8 s.

52
Solution

(a) Total distance covered by the giraffe after n leaps

$$= \frac{3(1-0.98^n)}{1-0.98}$$

$$= 150(1-0.98^n)$$

(b) Total distance covered by the cheetah after n leaps

$$= \frac{n}{2}[2(6)+(n-1)(-0.1)]$$

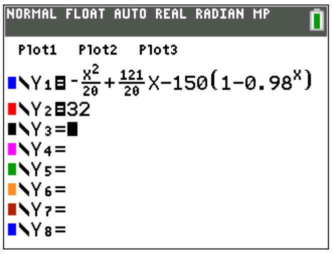
$$= 6n - \frac{1}{20}n(n-1)$$

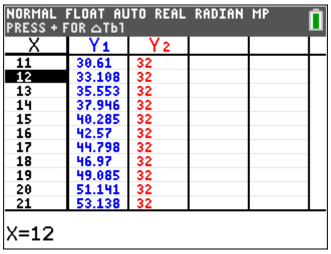
$$= -\frac{n^2}{20} + \frac{121}{20}n$$

(c) For the cheetah to catch the giraffe,

i.e. $$-\frac{n^2}{20} + \frac{121}{20}n - 150(1-0.98^n) \geq 32$$

From GC,

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X	Y1	Y2
11	30.61	32
12	33.108	32
13	35.553	32
14	37.946	32
15	40.285	32
16	42.57	32
17	44.798	32
18	46.97	32
19	49.085	32
20	51.141	32
21	53.138	32

X=12

Therefore, the number of leaps is 12.

(d) Distance covered by cheetah at the nth leap

$$= 6 + (n-1)(-0.1)$$

Distance covered by cheetah when it stops leaping $= 0$

i.e. $$6+(n-1)(-0.1) = 0$$

$$n = \frac{6+0.1}{0.1}$$

$$= 61$$

$\therefore$ the number of leaps taken before cheetah stops $= 60$

(e) Let the distance of the giraffe from the cheetah at the start be d m.

Giraffe will survive the chase if

$$150(1-0.98^{60}) + d > \frac{60}{2}[2(6)+59(-0.1)]$$

$$d > 77.63$$

The shortest distance is 77.7 m.

53
Solution

(a) 1^{st} set: $1 = 3^0 = 3^{1-1}$

2^{nd} set: $3 = 3^1 = 3^{2-1}$

3^{rd} set: $9 = 3^2 = 3^{3-1}$

By taking observation of the number of terms for each set nth set: 3^{n-1}

$\therefore$ number of integers in the nth set $= 3^{n-1}$

(b) Total number of integers from 1st set to the $(n-1)$th set

$=1+3+3^2+\ldots+3^{n-2}$

$=3^0+3^1+3^2+\ldots+3^{n-2}$ $\lhd$ use sum of first n terms of GP $= \dfrac{a(r^n-1)}{r-1}$

$= \dfrac{3^0(3^{n-1}-1)}{3-1}$ $\lhd$ where $a = 3^0$, $r = 3$ and $n = n-1$

$= \dfrac{3^{n-1}-1}{2}$

(c) First integer in the nth set

$=\left[\dfrac{3^{n-1}-1}{2}\right]$th term of the arithmetic progression

$=1+4\left[\dfrac{3^{n-1}-1}{2}+1-1\right]$ $\lhd$ use nth $= a+(n-1)d$, where $a=1$, $d=4$, $n=\dfrac{3^{n-1}-1}{2}$

$=2(3^{n-1})-1$

Last integer in the nth set

$=$ First integer in $(n+1)n$th set -4

$=2(3^n)-5$

(d) Sum of terms in the nth set

$=\dfrac{3^{n-1}}{2}\left[2(3^{n-1})-1+2(3^n)-5\right]$ $\lhd$ use sum of first n terms of AP $= \dfrac{n}{2}(a+l)$

$=3^{n-1}(3^{n-1}+3^n-3)$ $\lhd$ where $n =$ number of integers from 1st set to the $(n-1)$th set

$=3^n(3^{n-2}+3^{n-1}-1)$ $a =$ first integer in the nth set

$=3^n\left[4(3^{n-2})-1\right]$ $l =$ last integer in the nth set

(e) For sum of all the numbers in that set first exceeds 25 000

i.e. $n(2n^2+1) > 25\,000$

Using G.C, the least n is 5.

54
Solution

(a)

n	Length of n^{th} square
1	2
2	$\left(2^{\frac{1}{2}}\right)2 = 2^{\frac{3}{2}}$
3	$\left(2^{\frac{2}{2}}\right)2 = 2^{\frac{4}{2}}$
…	…

Refer to the above table and observe the number pattern.

The length of the nth square is $2^{\frac{n+1}{2}}$ mm.

(b) Given that a standard A4 paper measures 210 mm by 297 mm, then the side of the square < 210

i.e. $2^{\frac{n+1}{2}} < 210$

$$\frac{n+1}{2} < \frac{\ln(210)}{\ln 2}$$

$$n < 14.428$$

Hence maximum number of square is 14.

(c) From part **(a)**, length of the nth square is $2^{\frac{n+1}{2}}$.

Therefore, area of the nth square $= 2^{\frac{n+1}{2}} \times 2^{\frac{n+1}{2}} = 2^{n+1}$.

Area of the 1st square $= 2^2$ $\quad \triangleleft$ substitute $n = 1$ into 2^{n+1}

Area of the 4th square – Area of the 3rd square

$= 2^5 - 2^4$

$= 2^4(2-1)$

$= 2^4$

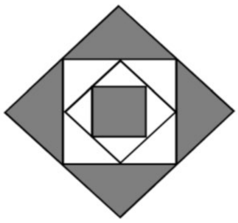


Figure 2

Total shaded area in Figure 2

(Area of the 4th square – Area of the 3rd square) + Area of the 1st square

$= 2^4 + 2^2$

$= 20$

(d) If 30 squares are drawn, then 28th square will be shaded.

Total shaded area

$= 2^2 + 2^4 + 2^7 + \ldots + 4^{28}$

$= 2^2 + \left(2^4 + 2^7 + \ldots + 4^{28}\right)$

$= 4 + \dfrac{2^4(2^{3(9)} - 1)}{(2^3 - 1)}$

$= 306,783,380 \text{ mm}^2$

$= 307 \text{ m}^2$ (correct to 3 s.f.)

n	Area of n^{th} square	Protruding area of n^{th} square
1	2^2	2^2
2	2^3	$2^3 - 2^2 = 2^2(2-1) = 2^2$
3	2^4	$2^4 - 2^3 = 2^3(2-1) = 2^3$
4	2^5	2^4
⋮	⋮	⋮
7	2^8	2^7
⋮	⋮	⋮
28	2^{29}	2^{28}

55
Solution

(a) Amount of drug before the 2nd dose

$$=\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}D\right)\right)$$

$$=\left(\frac{1}{2}\right)^3 D$$

$$=\frac{1}{8}D$$

(b)

n^{th} dose	Amount of drug $=U_n$
1	D
2	$\left(\frac{1}{2}\right)^3 D+D=\frac{1}{8}D+D$
3	$\left(\frac{1}{8}D+D\right)\left(\frac{1}{2}\right)^3+D$ $=\left(\frac{1}{8}\right)^2 D+\left(\frac{1}{8}\right)D+D$
...	...

Refer to the above table and observe the number pattern.

$$=\left(\frac{1}{8}\right)^{n-1} D+\left(\frac{1}{8}\right)^{n-2} D+\ldots+D$$

$$=\frac{D\left(1-\left(\frac{1}{8}\right)^n\right)}{1-\frac{1}{8}}$$

$$=\frac{8}{7}D\left(1-\left(\frac{1}{8}\right)^n\right)$$

$\therefore\ U_n=\frac{8}{7}D\left(1-\left(\frac{1}{8}\right)^n\right)$ (Shown), where $k=\frac{8}{7}$

(c) From **(b)** $U_n=\frac{8}{7}D\left(1-\left(\frac{1}{8}\right)^n\right)$

As $n\to\infty$, $\left(\frac{1}{8}\right)^n\to 0$.

Hence $U_n\to\frac{8}{7}D$.

The amount of drug that is present in the bloodstream if the patient continues to take it over a long period of time is $\frac{8}{7}D$ milligrams.

Given that Gabapentin toxicity develops when there is more than 60 mg of Gabapentin in the body

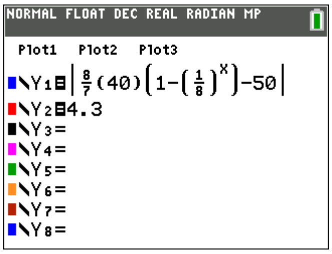
Thus, $\frac{8}{7}D\le 60$

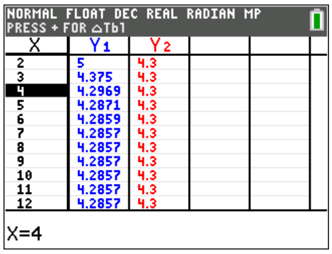
$D\le 52.5$

The largest possible value of D is 52.5

(d) Given that the amount of drug in the body after the last dose is within 4.3 mg of 50 mg,

i.e. $\left|\frac{8}{7}(40)\left(1-\left(\frac{1}{8}\right)^n\right)-50\right| \le 4.3$





Using GC

The least number of dose is 4.

Alternate Method

$$\left|\frac{8}{7}(40)\left(1-\left(\frac{1}{8}\right)^n\right)-50\right| \le 4.3$$

$$\left|\frac{320}{7}\left(1-\left(\frac{1}{8}\right)^n\right)-50\right| \le 4.3$$

$$\left|-\frac{30}{7}-\frac{320}{7}\left(\frac{1}{8}\right)^n\right| \le 4.3$$

$$\frac{30}{7}+\frac{320}{2}\left(\frac{1}{8}\right)^n \le 4.3$$

$$\left(\frac{1}{8}\right)^n \le 3.125\times10^{-4}$$

$$n \ge \frac{\ln(3.125\times10^{-4})}{\ln\left(\frac{1}{8}\right)}$$

$$n \ge 3.88$$

Least n is 4.

Therefore, the least number of dose is 4.

(e) Let t_n mg be the amount of drug taken by the patient on the nth day.

Given that on the first day, a dosage of 4 mg of the medication is taken and then increase the dosage by 2 mg each following day.

$\therefore$ the amount of drug taken by the patient on the nth day follows AP with first term 4 and common difference 2.

$\therefore\ t_n = 4+(n-1)(2).$

Consider the patient takes at most 20 mg,

$$t_n \le 20$$

$$4+(n-1)(2) \le 20$$

$$n \le 9$$

From 10th day onward, the patient has started to take 20 mg daily so that his doage will not exceed 20 mg.

Total amount of the drug the patient has taken over the entire 14-day course

$= \frac{9}{2}[2(4)+8(2)]+20\times5$

$= 208$ mg

Exercise 5

G Applications III (Advanced)

56
Solution

(a)

nth dollar	Accumulated points on Discover credit card
1	50
2	2(50)
...	
10	10(50)
11	$10(50)+55$
12	$10(50)+55+(55+5)$
13	$10(50)+55+(55+5)+\left[55+2(5)\right]$
14	$10(50)+55+(55+5)+\left[55+2(5)\right]+\left[55+3(5)\right]$
…	

If $n \le 10$
Number of points awarded $= 50n$

If $n \ge 11$
Number of points awarded

$$10(50)+55+(55+5)+\ldots+\left[55+(n-10-1)(5)\right]$$

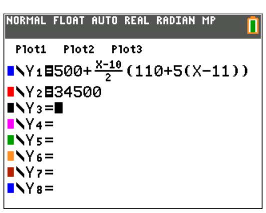
$$=500+\frac{n-10}{2}\left[2(55)+5(n-10-1)\right]$$

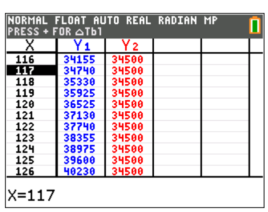
$$=500+\frac{n-10}{2}[110+5(n-11)],\ n\ge 11,\ n\in\mathbb{Z}$$

(b) Daniel already has accumulated 500 points. He only requires to accumulate another 34500 points to redeem a gift worth 35 000 points.

$$\therefore\quad 500+\frac{n-10}{2}[110+5(n-11)]\ge 35000-500$$

From G.C.,





The minimum amount of money that Daniel needs to charge on a *Discover* credit card to redeem a gift which costs 35000 points is \$117.

(c)

nth dollar	Accumulated points on Standard credit card
1	20
2	$20+20(1.05)$
3	$20+20(1.05)+20(1.05)^2$
…	

Number of points rewarded a customer charged \$$n$ to the *Standard* credit card

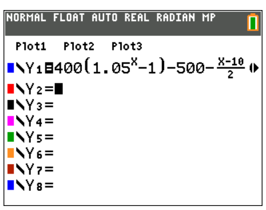
$$20+20(1.05)+20(1.05)^2+\ldots+20(1.05)^{n-1}$$
$$=20\left[1+1.05+1.05^2+\ldots+1.05^{n-1}\right]$$
$$=20\left[\frac{1.05^n-1}{1.05-1}\right]$$
$$=400(1.05^n-1)$$

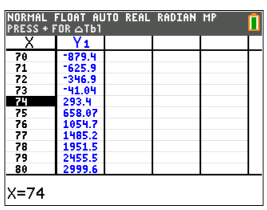
(d) For the number of point on the *Standard* card to be more points than the *Discover* card

i.e. Points on Standard credit card > Points on *Discover* credit card

$$400(1.05^n-1)>500+\frac{n-10}{2}\left[110+5(n-11)\right]$$
$$400(1.05^n-1)-500-\frac{n-10}{2}\left[110+5(n-11)\right]>0$$

From G.C.,





From the table, the minimum value of n is 74.

$\therefore$ the minimum amount to be charged on the Standard credit card is \$74.

57
Solution

(a) For $n \le 15$,

$$\mathrm{f}(n) = \frac{n}{2}[2(9500)+(n-1)400]$$
$$= 9500n + 200n(n-1)$$
$$= 200n^2 + 9600n$$

For $n > 15$,

$$\mathrm{f}(n) = 200(15^2) + 9300(15) + (n-15)15100$$
$$= 15100n - 42000$$

$$\mathrm{f}(n) = \begin{cases} 200n^2 + 9300n, & n \le 15, \\ 15100n - 42000, & n > 15, \end{cases}$$

where $A = -4200$ (Shown)

(b) Total annual wages earned by Sam if he works for Company P for 12 years

$$= \frac{9500\left[\left(1+\dfrac{r}{100}\right)^{12} - 1\right]}{1+\dfrac{r}{100} - r} \text{ .... (1)}$$

$\triangleleft$ use sum of GP: $\dfrac{a(1-r^n)}{1-r}$, where $a = 900$, $n = 12$ and common ratio $= 1+\dfrac{r}{100}$

Total annual wages earned by David if he works for Company Q for 15 years

$= 200(15^2) + 9300(15)$ $\triangleleft$ substitute $n = 15$ into $\mathrm{f}(n) = 15100n - 42000$ in **(a)**

$= 18450$ .... (2)

Equating (1) and (2)

$$\frac{9500\left[\left(1+\dfrac{r}{100}\right)^{12} - 1\right]}{1+\dfrac{r}{100} - r} = 18450$$

$$\frac{9500}{r}\left[\left(1+\frac{r}{100}\right)^{12} - 1\right] = 1845$$

From GC, $r = 8.39$ (correct to 2 dp)

$\therefore$ the value of r is 8.39.

58
Solution

(a)(i) Number of revolutions in nth minute

$$u_n = S_n - S_{n-1}$$
$$= 54n(29-n) - 54(n-1)(29-n+1)$$
$$= 1620 - 108n$$

$u_{n-1} = 1620 - 108(n-1)$ ⊲ replace n by $n-1$ in $54n(29-n)$

$$u_n - u_{n-1}$$
$$= 1620 - 108n - \left[1620 - 108(n-1)\right]$$
$$= -108$$

Since $u_n - u_{n-1} = -108$ is a constant (independent of n), the number of revolutions made in each minute follows an arithmetic progression.

(ii) When the car comes to a complete stop, it implies that number of revolutions at the instant becomes zero.

i.e. $u_n \le 0$

$$1620 - 108n \le 0$$
$$n \ge 15$$

Total number of revolutions

$$S_{15} = 54(15)(29-15)$$
$$= 11340$$

Distance travelled

= Total number of revolutions $\times \pi \times$ diameter of wheel

$= 11340 \times \pi \times 61$ cm

$= 21732165$ cm

$= 21.7$ km (to 3 s.f.)

The distance travelled by the car is 21.7 km.

(b)(i)

$$v_1 = (486+20)\left(\frac{2}{3}\right)$$

$$v_2 = \left[(486+20)\left(\frac{2}{3}\right)+20\right]\left(\frac{2}{3}\right)$$

$$= 486\left(\frac{2}{3}\right)^2 + 20\left(\frac{2}{3}\right)^2 + 20\left(\frac{2}{3}\right)$$

$$\vdots$$

$$v_n = 486\left(\frac{2}{3}\right)^n + 20\left(\frac{2}{3}\right)^n + 20\left(\frac{2}{3}\right)^{n-1} + \ldots + 20\left(\frac{2}{3}\right)$$

$$= 486\left(\frac{2}{3}\right)^n + 20\left[\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\left(\frac{2}{3}\right)}\right]$$

$$= 486\left(\frac{2}{3}\right)^n + 40\left(1-\left(\frac{2}{3}\right)^n\right)$$

$$= 446\left(\frac{2}{3}\right)^n + 40 \quad \text{(Shown)}$$

(ii) Since $\left(\frac{2}{3}\right)^n > 0$ for all $n > 0$, $v_n = 446\left(\frac{2}{3}\right)^n + 40 > 40$.

Thus, the wheel always rotates at a rate of more than 40 rpm.

(iii) $$446\left(\frac{2}{3}\right)^m + 40 < 45$$

$$\left(\frac{2}{3}\right)^m < \frac{5}{446}$$

$$m > \ln\frac{5}{446} \div \ln\frac{2}{3}$$

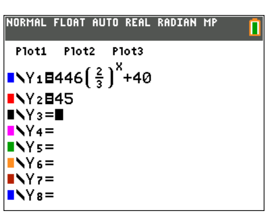
$$m > 11.1$$

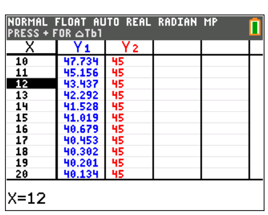
The least integer value of m is 12.

Alternative method

$$446\left(\frac{2}{3}\right)^m + 40 < 45$$

From the GC,





The least integer value of m is 12.

59
Solution

(a) Distance covered between 0 and 1st turn $= 100$

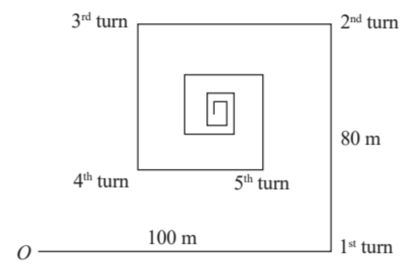
Distance covered between 1st and 2nd turn $= 80 = 100(0.8)$

Distance covered between 2nd and 3rd turn $= 100(0.8)^2$

Distance covered between 3rd and 4th turn $= 100(0.8)^2$

$\therefore$ the distance covered between $(n-1)$th and (n)th follows GP sequence.

First term $= 100$, ratio $= 0.8$



Distance covered between 11th and 12th turn

$= 100(0.8)^{12-1}$ $\quad \triangleleft$ where $n = 12$

$= 8.59$ m

The distance covered by the robot between the 11th turn and the 12th turn is 8.59 m.

(b) Let n be the number of turns.

Given that the robot has made after covering a total distance of 484 metres after making n number of turns,

i.e. $\quad 100 + 100(0.8) + 100(0.8)^2 + \ldots + 100(0.8)^{n-1} = 485$

$$\frac{100\left[1-0.8^n\right]}{1-0.8} = 485$$

$$1 - 0.8^n = 0.97$$

$$0.8^n = 0.03$$

$$n = \frac{\ln 0.03}{\ln 0.8}$$

$$n = 15.7$$

The robot has made 15 turns after covering a total distance of 485 m.

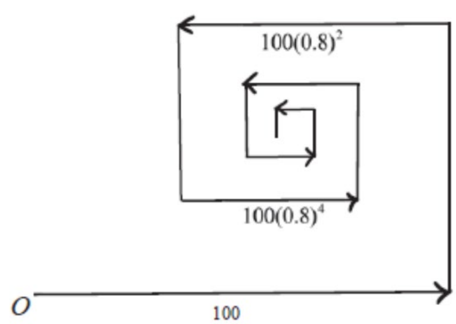
(c) Theoretical horizontal displacement

$= 100 - 100(0.8)^2 + 100(0.8)^4 - 100(0.8)^6 + \ldots$

$= \dfrac{100}{1-\left[-(0.8)^2\right]}$

$= 60.976$

$= 61.0$ m (correct 3 sf)



Theoretical vertical displacement

$= 100(0.8) - 100(0.8)^3 + 100(0.8)^5 - \ldots$

$= \dfrac{100(0.8)}{1-\left[-(0.8)^2\right]}$

$= 48.8$ m (correct 3 sf)

The theoretical final position that robot ends at 61.0 m due east and 48.8 m due north of O.

(d) The initial distance covered by the robot is 100 m. $\therefore$ the first term is 100.

The robot covers a total distance of 500 m just before making its 9th turn,

$\therefore$ total distance covered after 8 turns = 900

$$100\left(1-\frac{x}{100}\right)^{0}+100\left(1-\frac{x}{100}\right)^{1}+100\left(1-\frac{x}{100}\right)^{2}+\ldots+100\left(1-\frac{x}{100}\right)^{8}=900$$

The above series forms a GP. Let common ratio $r=1-\dfrac{x}{100}$, first term $100\left(1-\dfrac{x}{100}\right)^{0}$

$$\frac{100(1-r^{9})}{1-r}=500$$

$$r^{9}-5r+4=0$$

Using GC, $r=-1.30$ (rejected), $r=1$ (rejected), $r=0.84300178$

Thus, $1-\dfrac{x}{100}=0.84300178$

$$x=15.7$$

The value of x is 1.37.

Exercise 5

H Higher Order Questions

60
Solution

(a) Given $H = u_1 + u_2 + ... + u_n$

Series H is a geometric progression with common ratio r.

$$\therefore \ H = \frac{a(r^n - 1)}{r - 1}$$

$$I = \frac{1}{u_1} + \frac{1}{u_2} + ... + \frac{1}{u_n}$$

Series I is a a geometric progression with common ratio $\frac{1}{r}$

$$\therefore \quad I = \frac{\frac{1}{a}\left(1 - \frac{1}{r^n}\right)}{1 - \frac{1}{r}}$$

$$= \frac{1}{ar^{n-1}} \frac{r^n - 1}{r - 1}$$

$$\frac{H}{I} = \frac{a(r^n - 1)}{r - 1} \times \frac{ar^{n-1}(r - 1)}{r^n - 1}$$

$$= a \times ar^{n-1}$$

$= u_1 u_n$ (Proved) .... (1)

(b) $u_1 \times u_2 \times u_3 \times ... \times u_n$

$$= a \times (ar) \times (ar^2) \times (ar^3) \times ... \times (ar^{n-1})$$

$$= a^n (r^{1+2+3+...+(n-1)})$$

$$= a^n r^{\frac{(n-1)n}{2}}$$

From (1): $\frac{H}{C} = u_1 u_n$

$$= a \times ar^{n-1}$$

$$= a^2 r^{n-1}$$

Since $u_1 u_n = a^2 r^{n-1}$

$$u_1 u_2 \ ... \ u_n = a^n r^{\frac{(n-1)n}{2}}$$

$$= (a^2 r^{n-1})^{\frac{n}{2}}$$

$$= \left(\frac{H}{I}\right)^{\frac{n}{2}}$$

61
Solution

Given that the terms $\dfrac{1}{u_{n+1}-u_n}$, $\dfrac{1}{2u_{n+1}}$ and $\dfrac{1}{u_{n+1}-u_{n+2}}$ are consecutive terms of an arithmetic progression, the common difference can be either $\dfrac{1}{2u_{n+1}}-\dfrac{1}{u_{n+1}-u_n}$ or $\dfrac{1}{2u_{n+1}}-\dfrac{1}{u_{n+1}-u_n}$.

$$\therefore \quad \frac{1}{2u_{n+1}}-\frac{1}{u_{n+1}-u_n}=\frac{1}{u_{n+1}-u_{n+2}}-\frac{1}{2u_{n+1}}$$

$$\frac{u_{n+1}-u_n-2u_{n+1}}{2u_{n+1}(u_{n+1}-u_n)}=\frac{2u_{n+1}-u_{n+1}+u_{n+2}}{2u_{n+1}(u_{n+1}-u_{n+2})}$$

$$\frac{-u_n-u_{n+1}}{\cancel{2u_{n+1}}(u_{n+1}-u_n)}=\frac{u_{n+1}+u_{n+2}}{\cancel{2u_{n+1}}(u_{n+1}-u_{n+2})}$$

$$(-u_n-u_{n+1})(u_{n+1}-u_{n+2})=(u_{n+1}+u_{n+2})(u_{n+1}-u_n)$$

$$-u_nu_{n+1}+u_nu_{n+2}-(u_{n+1})^2+u_{n+1}u_{n+2}=(u_{n+1})^2-u_nu_{n+1}+u_{n+1}u_{n+2}-u_nu_{n+2}$$

$$2u_nu_{n+2}=2(u_{n+1})^2$$

$$u_nu_{n+2}=(u_{n+1})^2$$

$$\frac{u_{n+2}}{u_{n+1}}=\frac{u_{n+1}}{u_n}$$

Therefore $\{u_n\}$ forms a geometric progression.

62
Solution

(a) $a_2 = 3.5a_1 - 1.2$

$a_3 = 3.5a_2 - 1.2$

$= 3.5(3.5a_1 - 1.2) - 1.2$

$= 3.5^2 a_1 - 1.2(3.5) - 1.2$

$= 3.5^2 a_1 - 1.2(1 + 3.5)$

$a_4 = 3.5\left[3.5^2 a_1 - 1.2(1 + 3.5)\right] - 1.2$

$= 3.5^3 a_1 - 1.2(1 + 3.5 + 3.5^2)$

$a_n = 3.5^{n-1} a_1 - 1.2(1 + 3.5 + 3.5^2 + \ldots + 3.5^{n-2})$

$a_n = 3.5^{n-1} a_1 - 1.2(3.5^0 + 3.5^1 + 3.5^2 + \ldots + 3.5^{n-2})$ $\lhd\ 3.5^0 + 3.5^1 + 3.5^2 + \ldots + 3.5^{n-2}$ forms geometric series

$= 3.5^{n-1} a_1 - 1.2\left(\dfrac{(1)(3.5^{n-1} - 1)}{3.5 - 1}\right)$

$= 3.5^{n-1} a_1 - 0.48(3.5^{n-1} - 1)$ (Shown)

(b) Number of microbes in Lab *A* at the end of the third week, before the microbes are transferred to the research centre

$= a_4 + 1.2$

$= 3.5^{4-1}(0.5) - 0.48(3.5^{4-1} - 1) + 1.2$

$= 2.5375$

Number of microbes in Lab *A* at the end of the third week before transferred is 2.5375 millions

(c) Let d_n be the number of cells (in millions) in Lab *B* at the start of the *n*th week.

$d_1 = 0$

$d_2 = 0.5$

$d_3 = 0.5 + 5$

$d_4 = 0.5 + 5 + (5 + 5)$

$= 0.5 + 5 + 2(5)$

By observation,

$d_n = 0.5 + (1)5 + 2(5) + 3(5) + \ldots + (n-2)(5)$

$= 0.5 + \dfrac{n-2}{2}(10 + (n - 2 - 1)(5))$

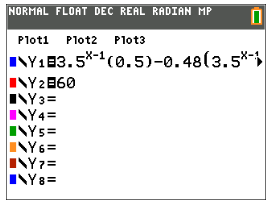
$= 0.5 + \dfrac{n-2}{2}(10 + 5(n - 3))$

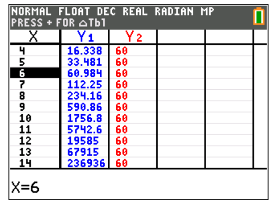
For the research project to begin, number of microbes must need at least 60 million microbes,

i.e. number of microbes in Lab A + number of microbes in Lab $B \geq 60$

$$3.5^{n-1}(0.5) - 0.48(3.5^{n-1} - 1) + 0.5 + \frac{n-2}{2}(10 + 5(n-3)) \geq 60$$

By G.C., least $n = 6$





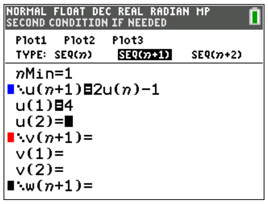
$\therefore$ the earliest week is week 6.

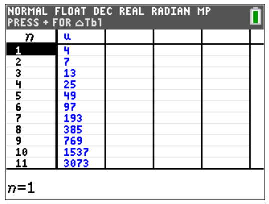
63
Solution

(a)(i) When $a = 4$ and $k = 1$ such that $u_1 = 4$

$\therefore \quad u_{n+1} = 2u_n - 1$

Using GC



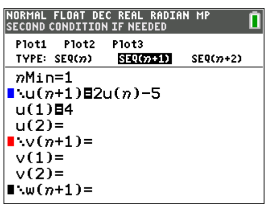


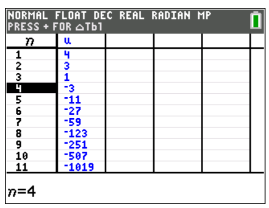
The population is increasing. The population will grow to infinity.

When $a = 4$ and $k = 5$ such that $u_1 = 4$

$\therefore \quad u_{n+1} = 2u_n - 5$

Using GC





The population is increasing. The population will become extinct in 4th year.

Alternative Method

When $a = 4$ and $k = 5$ such that $u_1 = 4$

$\therefore \ u_{n+1} = 2u_n - 5$

When $n = 1$, $\quad u_2 = 2(4) - 5 = 3$

When $n = 2$, $\quad u_3 = 2(3) - 5 = 1$

When $n = 3$ $\quad u_4 = 2(1) - 5 = -3 < 0$

The population will become extinct in 4th year.

(a)(ii) Given there are 52 animals in the 3rd year, i.e. $u_3 = 52$

Using $u_{n+1} = 2u_n - k$ ◁ substituting $n = 1$

$u_2 = 2u_1 - k$ ◁ substituting $u_1 = 4$

$= 2(4) - k$

$= 8 - k$

$u_{n+1} = 2u_n - k$ ◁ substituting $n = 2$

$u_3 = 2u_2 - k$ ◁ substituting $u_2 = 8 - k$

$u_3 = 2(8-k) - k$

$52 = 16 - 3k$ ◁ given $u_3 = 52$

$k = -12$

$\therefore\ k = -12$

(b) Given $v_n = u_n - k$ .... (1)

Replace n by $n+1$ in (1): $v_{n+1} = u_{n+1} - k$

Take $\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1} - k}{u_n - k}$ ◁ from **(a)** $u_{n+1} = 2u_n - k$

$= \dfrac{(2u_n - k) - k}{u_n - k}$

$= 2$ (a constant)

Thus v_n is a geometric progression

Already verified that v_n is GP.

$\therefore$ the first term $= v_1$ and common ratio $= 2$

$v_n = v_1(2)^{n-1}$ ◁ use nth term of GP: nth $= ar^{n-1}$

$= (u_1 - k)2^{n-1}$ ◁ from **(b)** $v_n = u_n - k$ and substitute $n = 1$

$= (a-k)2^{n-1}$ ◁ $u_1 = a$ (given)

$v_n + k = (a-k)2^{n-1} + k$ ◁ add k on both sides

$u_n = (a-k)2^{n-1} + k$ ◁ from **(b)** $v_n = u_n - k$. $\therefore\ v_n + k = u_n$

$u_{n+1} = (a-k)2^n + k$ ◁ replace n by $n-1$ (Shown)

(c) From **(b)** $u_{n+1} = (a-k)2^n + k$

As $n \to \infty,\ 2^n \to \infty$

For population to stay stabilized as $n \to \infty$.

$a - k = 0$

$a = k$

The value of a is k.

(d) From **(b)**: $u_{n+1} = (a-k)2^n + k$

Replace $n+1$ by $n-1$

$$u_n = (a-k)2^{n-1} + k \text{ .... } (2)$$

From the 5th year and ending at the 80th year at every 5-year interval, there are $\frac{(80-5)}{5} + 1 = 16$ terms.

Sum of the population

$= u_5 + u_{10} + u_{15} + ... + u_{75} + +u_{80}$

$= \left[(a-k)2^{5-1} + k\right] + \left[(a-k)2^{10-1} + k\right] + \left[(a-k)2^{20-1} + k\right] + ... + \left[(a-k)2^{75-1} + k\right] + \left[(a-k)2^{80-1} + k\right]$

$= (a-k)[2^4 + 2^9 + ... + 2^{79}] + 16k$

$= \frac{2^4[(2^5)^{16} - 1]}{2^5 - 1}(a-k) + 16k$

$= \frac{16}{31}[2^{80} - 1](a-k) + 16k$

Sum of the population at every 5-year interval, starting from the 5th year and ending at the 80th year is $\frac{16}{31}[2^{80} - 1](a-k) + 16k$

Note

The series $2^4 + 2^9 + ... + 2^{79}$ follows geometric series with first term 2^4, common ratio 2^5 and number of terms 16.

64
Solution

Terms in the sequence: $p, p, p+d, pr, p+2d, pr^2, ..., p+(2n-1)d, pr^{2n-1}$

Odd terms: $p, p+d, p+2d, ..., (2n-1)d$

Even terms: $p, pr, pr^2, ..., pr^{2n-1}$

Sum of odd terms

$= \frac{n}{2}\left[2p+(n-1)p\right]$ ⊲ use sum of AP with first term p, common difference p and number of terms n

$= \frac{pn}{2}(n+1)$

Sum of even terms

$= \frac{p(1.2^n - 1)}{1.2-1}$ ⊲ use sum of GP with first term p, common ratio 1.2 and number of terms n

$= 5p(1.2^n - 1)$

Given that the sum of all the odd terms is less than the sum all of the even terms,

i.e. $\frac{pn}{2}(n+1) < 5p(1.2^n - 1)$

$\frac{n}{2}(n+1) < 5(1.2^n - 1)$ (since $p > 0$)

Using GC, the least value of n is 22.

65
Solution

(a) Since rate of elimination is $\frac{\alpha}{100}$, the amount of medication that remains in the bloodstream just before the $(n+1)$th injection

$=\left(1-\frac{\alpha}{100}\right)$ of the amount of medication in bloodstream immediately after the nth injection.

Given $x=\left(1-\frac{\alpha}{100}\right)$

Number of injections	Amount of medication after nth injection	Amount of medication just before $(n+1)$th injection
1	6	$6x$
2	$6x+4$	$x(6x+4)=6x^2+4x$
3	$x(6x^2+4x)+4=6x^3+4x^2+4$	$x(6x^3+4x^2+4)=6x^4+4x^3+4x$

Refer to above.

Amount of medication remaining immediately just before 3rd injection

$=6x^2+4x$ units (Shown)

(b) Amount of medication remaining in the patients's body after the 3rd injection is

$=x(6x+4)+4$

$=6x^2+4x+4$, where $x=1-\frac{a}{100}$.

(c)(i) Amount of medication after 1st injection $=6$

Amount of medication after 2nd injection $=6x+4$

Amount of medication after 3rd injection $=6x^2+4x+4$

Amount of medication after 4th injection $=6x^3+4x^2+4x+4$

Amount of medication after nth injection

$=6x^{n-1}+4x^{n}+...+4x^2+4x^1+4x^0$, for $1\le n\le 30$

$=6x^{n-1}+\left(4x^0+4x^1+4x^2+...+4x^n\right)$ ◁ The series forms an GP with first term $4x^0$, common ratio x and number of terms $n-1$

$=6x^{n-1}+4x\left(\frac{1-x^{29}}{1-x}\right)$

$\therefore$ $\mathrm{Y}(n)=6x^{n-1}+4\left(\frac{1-x^{n-1}}{1-x}\right)$, where $1\le n\le 30$ (Shown)

(c)(ii) $\mathrm{f}(30) = \mathrm{Y}(30) = 6x^{29} + 4\left(\dfrac{1-x^{29}}{1-x}\right)$

Since the injection on the 30th day is the last injection,

$$\mathrm{f}(31) = \mathrm{g}(31) = x\left[6x^{29} + 4\left(\frac{1-x^{29}}{1-x}\right)\right] = 6x^{30} + 4x\left(\frac{1-x^{29}}{1-x}\right)$$

$$\mathrm{f}(32) = \mathrm{g}(32) = x^2\left[6x^{29} + 4\left(\frac{1-x^{29}}{1-x}\right)\right] = 6x^{31} + 4x^2\left(\frac{1-x^{29}}{1-x}\right)$$

$$\mathrm{f}(33) = \mathrm{g}(33) = x^3\left[6x^{29} + 4\left(\frac{1-x^{29}}{1-x}\right)\right] = 6x^{32} + 4x^3\left(\frac{1-x^{29}}{1-x}\right)$$

$\therefore$ For $n \geq 31$,

$$\mathrm{g}(n) = x^{n-3}\left[6x^{29} + 4\left(\frac{1-x^{29}}{1-x}\right)\right]$$

$$= 6x^{n-1} + 4x^{n-30}\left(\frac{1-x^{29}}{1-x}\right)$$

(c)(iii) Since the last day of injection is on the 30th day, to find the amount of medication in the long run,

we use $\mathrm{g}(n) = 6x^{n-1} + 4x^{n-30}\left(\dfrac{1-x^{29}}{1-x}\right)$, since $n > 30$.

As $n \to \infty$ and as $0 < x = 1 - \dfrac{\alpha}{100} < 1$,

$6x^{n-1} \to 0$ and $4x^{n-30} \to 0$

$\therefore\ 6x^{n-1} + 4x^{n-30}\left(\dfrac{1-x^{29}}{1-x}\right) \to 0.$

So, $\mathrm{g}(n) \to 0$

Thus the amount of medication in the patient's blood stream in the long run will decrease and approaches to 0.

(d) Given that the medication elimination rate is 20%, i.e. $\alpha = 20$

$\therefore\ x = 1 - \dfrac{20}{100} = 0.8.$

Substitute $x = 0.8$ into $\mathrm{f}(n)$.

$$\mathrm{f}(n) = \begin{cases} \mathrm{Y}(n) = 6(0.8)^{n-1} + 4\left(\dfrac{1-0.8^{n-1}}{1-0.8}\right), & \text{for } 1 \leq n \leq 30 \\ \mathrm{g}(n) = 6(0.8)^{n-1} + 4(0.8)^{n-30}\left(\dfrac{1-0.8^{29}}{1-0.8}\right), & \text{for } n \geq 31 \end{cases}$$

When $n = 30$,

$$\mathrm{Y}(30) = 6(0.8)^{30-1} + 4\left(\frac{1-0.8^{30-1}}{1-0.8}\right)$$

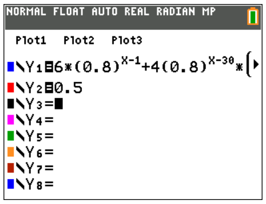
$$= 19.978 \text{ units}$$

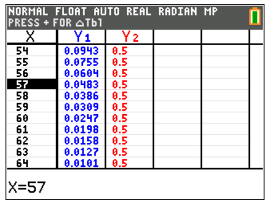
Note that $\mathrm{f}(30) = 19.978$ units. That is to say, on 30th day amount of codeine remaining in the patient's bloodstream is 19.978 units. However we need to find the day that the amount of codeine remaining in the patient's bloodstream falls to 0.05 units or less. Thus we should use $\mathrm{g}(n) = 6(0.8)^{n-1} + 4(0.8)^{n-30}\left(\dfrac{1-0.8^{29}}{1-0.8}\right)$, for $n > 30$.

Now consider

$$6(0.8)^{n-1}+4(0.8)^{n-30}\left(\frac{1-0.8^{29}}{1-0.8}\right)<0.05$$

Using GC





The least number of days for the residual amount of medication to fall below 0.05 units is 57.

Alternative method (analytical method)

$$6(0.8)^{n-1}+4(0.8)^{n-30}\left(\frac{1-0.8^{29}}{1-0.8}\right)<0.05$$

$$(0.8)^{n-30}\left(6(0.8)^{29}+4\left(\frac{1-0.8^{29}}{1-0.8}\right)\right)<0.05$$

$$(0.8)^{n-30}<\frac{0.05}{6(0.8)^{29}+4\left(\frac{1-0.8^{29}}{1-0.8}\right)}$$

$$(n-30)\ln 0.8<\ln\left(\frac{0.05}{6(0.8)^{29}+4\left(\frac{1-0.8^{29}}{1-0.8}\right)}\right)$$

$$n-30>\frac{\ln\left(\frac{0.05}{6(0.8)^{29}+4\left(\frac{1-0.8^{29}}{1-0.8}\right)}\right)}{\ln 0.8}$$

$$n>56.8$$

$\therefore$ the least $n=57$

66
Solution

(a)(i) Let n be the number of years beginning from 1 Jan 2000, i.e. $n=1$ on 31 Dec 2000.

n	Year	Growth in height in a year	Actual height (measured on 31 Dec)
1	1999	2.2	44
2	2000	$2.2-b$	$44+(2.2-b)$
3	2001	$2.2-2b$	$44+(2.2-b)+(2.2-2b)$
…	…		…
11	2010	$2.2-11b$	60
nth		$2.2-nb$	$44+(2.2-b)+...+(2.2-nb)$

Refer to the 3rd column
The change in height of the tree in each year forms an arithmetic progression with first term $2.2-b$ (year 2000) and common difference $-b$.

For 31 Dec 2010, i.e. the value of $n=11$
Let U_n be the growth in the height of the Meranti tree in nth year, from 31 Dec 2000

$$\therefore \quad U_1 = 2.2-b$$
$$U_{11} = 2.2-11b$$

Growth in the height of the Meranti tree from 31 Dec 2000 to 31 Dec 2010
$=60-44$
$=16$

Use sum of AP formuale

$$\frac{11}{2}(U_1+U_{11})=16$$
$$\frac{11}{2}(2.2-b+2.2-11b)=16$$
$$b=0.12424$$
$$=0.124 \text{ (3s.f)} \quad \text{(Shown)}$$

Growth in the height of the Meranti tree in 2001, $U_{11}=2.2-2b$
When $b=0.124$

$$U_{11}=2.2-2(0.124)$$
$$=1.952$$

Growth in height of the Meranti tree in the year 2001 is 1.952 m.

(ii) $U_1 + (n-1)(-b) < 0.125$

$(2.2 - 0.12424) + (n-1)(-0.12424) < 0.125$

$2.2 - n(0.12424) < 0.125$

$n > 16.7$

Since $n \in \mathbb{Z}^+$, least $n = 17$.

The tree reaches its maximum height in 2016 (i.e. 17th year starting from 2000).

Height of tree in 2016

$= 44 + (2.2 - b) + (2.2 - 2b) + ... + (2.2 - 17b)$

$= 44 + 2.2 \times 17 - (b + 2b + ... + 17b)$

When $b = 0.12424$

$= 62.4$

$\therefore$ the maximum height of the tree is 62.4m.

(b)

Year	Growth (measured on 31 Dec)	Length (measured on 31 Dec)
2000	6	40
2001	$6k$	
…	…	…
2004	$6k^4$	55

The growth in length of roots in each year forms a geometric progression with first term $6k$ (year 2001) and common ratio k.

Growth in length of roots measured on 31 December 2004

$= \dfrac{6k(1-k^4)}{1-k}$

Given that meranti tree was measured at 40 metres on 31 December 2000 and 55 metres on 31 December 2004

$\therefore$ the growth in length of roots of the Meranti tree from 31 December 2000 to 31 December 2004 $= 55 - 40 = 15$

$\therefore \quad \dfrac{6k(1-k^4)}{1-k} = 15$

$6k(1-k^4) = 15(1-k)$

$-6k^5 + 21k - 15 = 0$

From GC, $k = 0.82059 = 0.821$ (correct to 3s.f)

Maximum length

$= 40 + \dfrac{6(0.82059)}{1 - 0.82059}$ ⊲ use sum to infinity, where the first term 6 and common ratio 0.82059

$= 67.443$

$= 67.4$ (correct to 3 s.f)

$\therefore$ the maximum length that the roots can grow is 67.4 m.

67
Solution

(a) $A_1 = 80000 + 80000 \times 0.02 \times \dfrac{1}{4} - R$

$= 80000(1.005) - R$

$A_2 = (A_1) \times 1.005 - R$

$= [80000(1.005) - R] \times 1.005 - R$

$= 80000(1.005)^2 - 1.005R - R$

(b) From **(a)**: $A_1 = 80000(1.005) - R$

$A_2 = 80000(1.005)^2 - 1.005R - R$

From observations,

$A_3 = 80000(1.005)^3 - R(1 + 1.005 + 1.005^2)$

$A_4 = 80000(1.005)^4 - R(1 + 1.005 + 1.005^2 + 1.005^3)$

$\vdots$

$\therefore \quad A_{4m} = 80000(1.005)^{4m} - R(1 + 1.005 + 1.005^2 + ... + 1.005^{4m-1})$

Repayment amount made at the end of the $4m$ th quarter

$= 80000(1.005)^{4m} - R(1 + 1.005 + 1.005^2 + ... + 1.005^{4m-1})$

$= 80000(1.005)^{4m} - R(1.005^0 + 1.005^1 + 1.005^2 + ... + 1.005^{4m-1})$

$= 80000(1.005)^{4m} - R\left[\dfrac{1.005^{4m} - 1}{1.005 - 1}\right]$

$= 80000(1.005)^{4m} - 200R(1.005^{4m} - 1)$

$\therefore \; A_{4m} = 80000(1.005)^{4m} - 200R(1.005^{4m} - 1)$

Given that the loan is fully repaid at the mth year, i.e. $A_{4m} = 0$

$80000(1.005)^{4m} - 200R(1.005^{4m} - 1) = 0$

$200R(1.005^{4m} - 1) = 80000(1.005)^{4m}$

$R = \dfrac{80000(1.005)^{4m}}{200(1.005^{4m} - 1)}$

$= \dfrac{400(1.005)^{4m}}{1.005^{4m} - 1}$, where $B = 400$ and $C = 1.005$ .... (1)

(c) Given that the mortgage loan is over a 10-year period, i.e. $m = 10$.

Substitute $m = 10$ into (1)

$\therefore \quad R = \dfrac{400(1.005)^{40}}{1.005^{40} - 1}$

$= 2211.64$ (correct to 2 dp)

Alex's quarterly repayment is \$2211.64.

68
Solution

(a)

Day	Mass of Metformin right after treatment (mg)
1	A
2	$A(0.9)+B$
3	$A(0.9)^2+B(0.9)+B$
4	$A(0.9)^3+B(0.9)^2+B(0.9)+B$
$\cdots$	$\cdots$
n	$(0.9)^{n-1}+B(0.9)^{n-2}+B(0.9)^{n-3}+\ldots+B(0.9)+B$

Amount of metformin after nth treatment (in mg)

$=A(0.9)^{n-1}+B(0.9)^{n-2}+B(0.9)^{n-3}+\ldots+B(0.9)+B$ ⊲ refer to the last row of the table above

$=A(0.9)^{n-1}+\left[B(0.9)^{n-2}+B(0.9)^{n-3}+\ldots+B(0.9)^1+B(0.9)^0\right]$ ⊲ use sum of GP formulae

$$=A(0.9)^{n-1}+\frac{B\left[1-(0.9)^{n-1}\right]}{1-0.9}$$

$$=A(0.9)^{n-1}+10B\left[1-(0.9)^{n-1}\right]$$

$$=10B+(A-10B)(0.9)^{n-1} \text{ .... (1) (Shown)}$$

(b) From **(a)**: $10B+(A-10B)(0.9)^{n-1}$

As $n\to\infty$, $(A-10B)(0.9)^{n-1}\to 0$

In the long run, the quantity of metformin left in the body right after treatment will be $10B$, which is not affected by the value of A. $\therefore$ there is no residual of the initial dose in the body over time.

(c) Given that the amount dosage of metformin taking on the first day is 40 mg dose and then reduce the dosage to 18 mg per day, i.e., $A=40$ and $B=18$.

Substitute $A=40$ and $B=18$ into (1)

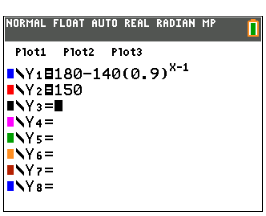
Amount of metformin after nth treatment $=10(18)+(40-10\times 18)(0.9)^{n-1}$

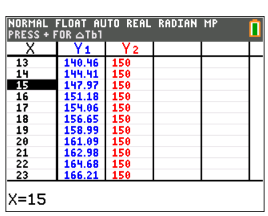
The treatment will stop before the body has absorbed more than 150 mg of metformin.

$$\therefore \quad 10(18)+(40-10\times 18)(0.9)^{n-1}>150$$

$$180-140(0.9)^{n-1}>150$$

Using GC





Therefore the maximum number of days this treatment can be carried out for is 15 days.

Alternative Method

$$180-140(0.9)^{n-1}>150$$
$$140(0.9)^{n-1}<30$$
$$(n-1)\ln 0.9<\ln\frac{3}{14}$$
$$n-1>\frac{\ln\frac{3}{14}}{\ln 0.9}$$
$$n>15.6$$

Therefore the maximum number of days this treatment can be carried out for is 15 days.

(d)

Day	Difference in Mass of Metformin remaining right after treatment (mg)
4	0
5	18
6	18(0.9)
7	$18(0.9)^2$
$\cdots$	$\cdots$
n	$18(0.9)^{n-5}$

When $n=15$,

Difference on Day $15=18(0.9)^{10}$

$=6.28$ mg (3 s.f.)

(e) Let $\$C_n$ be the cost of treatment for the whole nth year, starting from 2007.

C_n is an AP with common difference $=48$, first term $C_1=1440$

$$C_{16}=1440+(16-1)48 \quad \triangleleft \text{ cost of treatment for the whole year 2022}$$
$$=2160$$

Cost of treatment from Jan 2007 to Dec 2022

$$= \frac{n}{2}(C_1 + C_{16})$$

$$= \frac{16}{2}(1440 + 2160)$$

$$= 28800$$

Cost of treatment for Sandra

$$= 28800 - \frac{1}{2}C_1 - \frac{1}{2}C_{16}$$

$$= 28800 - 720 - 1080$$

$$= \$27000$$

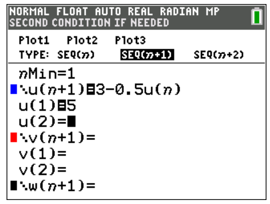
The total amount of money Sandra spent on the treatment plan from July 2007 to June 2022 is $27000.

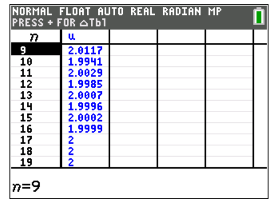
Exercise 10

I Higher Order Questions

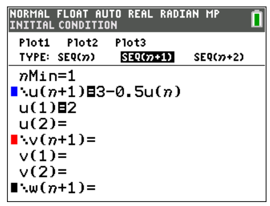
69
Solution

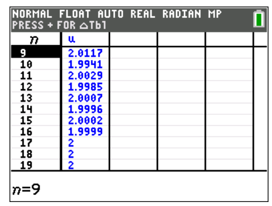
(a) When (A) $c = 5$
The sequence initialty decreases and subsequently alternates (increasing and decreasing) and eventually converges to 2.





When (B) $c = 2$
Every term in the sequence is a constant with a value of 2.





(b) Given $u_1 = c$

$u_2 = 3 - 0.5u_1 = 3 - 0.5c$

$u_3 = 3 - 0.5u_2 = 3 - 0.5(3 - 0.5c) = 1.5 + 0.25c$

Given $2u_3 = -5u_2$,

$$2(1.5 + 0.25c) = -5(3 - 0.5c)$$
$$3 + 0.5c = -15 + 2.5c$$
$$2c = 18$$
$$c = 9$$

Alternative

$$2(3 - 0.5u_2) = -5u_2$$
$$u_2 = -1.5$$
$$3 - 0.5u_1 = -1.5$$
$$2c = 18 \text{ (since } u_1 = c)$$
$$c = 9$$

70
Solution

(a) Given $S_n = an + \dfrac{b}{n+1} + c$ .... (1)

The first term of the sequence is $\dfrac{5}{2}$. i.e. $S_1 = \dfrac{5}{2}$

Substitute $n = 1$ into (1)

$$S_1 = a(1) + \frac{b}{1+1} + c$$

$$\therefore \qquad \frac{5}{2} = a + \frac{b}{2} + c \text{ .... (2)}$$

The second term of the sequence is $\dfrac{13}{6}$. i.e. $S_2 = \dfrac{5}{2}$

$$S_2 = a(2) + \frac{b}{2+1} + c$$

$$\frac{14}{3} = 2a + \frac{b}{3} + c \text{ .... (3)}$$

Given $\qquad S_5 = \dfrac{65}{6}$

$$5a + \frac{b}{6} + c = \frac{65}{6} \text{ .... (3)}$$

From GC, $a = 2, b = -1, c = 1$

Substitute $a = 2, b = -1, c = 1$ into (1)

Hence, $S_n = 2n - \dfrac{1}{n+1} + 1$

(b) $u_n = S_n - S_{n-1}$

$$= \left(2n - \frac{1}{n+1} + 1\right) - \left(2(n-1) - \frac{1}{n-1+1} + 1\right)$$

$$= 2n - \frac{1}{n+1} + 1 - 2n + 2 + \frac{1}{n} - 1$$

$$= 2 - \left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$= 2 - \left(\frac{n-(n+1)}{n(n+1)}\right)$$

$$= 2 + \frac{1}{n(n+1)}$$

(c) From **(b)**, $u_n = 2 + \dfrac{1}{n(n+1)}$

As n gets larger, $\dfrac{1}{n(n+1)}$ gets smaller. So $u_n > u_{n+1}$.

As $n \to \infty, \dfrac{1}{n(n+1)} \to 0, u_n \to 2.$

The sequence is decreasing $(u_n > u_{n+1})$ and converges to 2.

71
Solution

(a) Given $u_{n+2} = Au_{n+1} - u_n$ .... (1)

Substitute $n = 1$ into (1)

$$u_3 = Au_2 - u_1$$
$$5 = Au_2 - 1$$
$$u_2 = \frac{6}{A} \text{ .... (2)}$$

Substitute $n = 2$ into (2)

$$u_4 = Au_3 - u_2$$
$$13 = 5A - u_2 \text{ .... (3)}$$

Substitute (2) into (3)

$$13 = 5A - \frac{6}{A}$$
$$13A = 5A^2 - 6$$

$$\therefore \quad 5A^2 - 13A - 6 = 0$$
$$(A-3)(5A+2) = 0$$
$\therefore$ $A = 3$ (since $A > 0$)

Substitute $A = 3$ into (2)

$$\therefore \quad u_2 = \frac{6}{3}$$
$$u_2 = 2$$

(b)(i) $v_n = S_n - S_{n-1}$

$$= 2n^2 + 4n - [2(n-1)^2 + 4(n-1)]$$
$$= 2n^2 + 4n - 2(n^2 - 2n + 1) - 4(n-1)$$
$$= 2n^2 + 4n - 2n^2 + 4n - 2 - 4n + 4$$
$$= 4n + 2$$

(b)(ii) $v_2 + v_4 + v_6 + \ldots + v_{48}$

$$= [4(2)+2] + [4(4)+2] + [4(6)+2] + \ldots + [4(46)+2] + [4(48)+2]$$

$= \frac{24}{2}(10 + 194)$ ⊲ use sum of GP with first term 10, last term 194 and number of terms 24

$$= 2448$$

72
Solution

(a) Consider $\dfrac{T_n}{T_{n-1}} = \dfrac{e^{u_n}}{e^{u_{n-1}}}$

$= e^{u_n - u_{n-1}}$

Given that $\{u_n\}$ is an arithmetic progression, then $u_n - u_{n-1}$ gives a common difference d

$\therefore \quad \dfrac{T_n}{T_{n-1}} = e^d$

Since d is a constant, e^d is a constant independent of n.

Thus, $\{T_n\}$ is a geometric progression. (Shown)

If $\{u_n\}$ is a decreasing sequence, then the common difference must be negative.

$\therefore \quad d < 0$

$0 < e^d < 1$

$0 < r < 1$

For geometric progression converges, then $|r| < 1$

Since $0 < r < 1$, the sequence $\{T_n\}$ is a geometric progression. (Shown)

(b) $S_{12} = 2(S_{36} - S_{12})$

$3S_{12} = 2S_{36}$

$3\left[\dfrac{r^{12}-1}{r-1}\right] = 2\left[\dfrac{r^{36}-1}{r-1}\right]$

$3(r^{12}-1) = 2(r^{36}-1)$

$3r^{12} - 2r^{36} - 1 = 0$

Let $y = r^{12}$.

$-2y^3 + 3y - 1 = 0$

$(y-1)(-2y^2 - 2y + 1) = 0$

$y^2 + y - \dfrac{1}{2} = 0 \quad$ (since $y = r^{12} \neq 1$ as $0 < r < 1$)

$\left(y + \dfrac{1}{2}\right)^2 - \dfrac{1}{2} - \dfrac{1}{4} = 0 \quad$ or $\quad y = \dfrac{-1 \pm \sqrt{(-1)^2 - 4(1)\left(-\dfrac{1}{2}\right)}}{2}$

$\left(y + \dfrac{1}{2}\right)^2 = \dfrac{3}{4} \quad$ or $\quad y = \dfrac{-1 \pm \sqrt{1+2}}{2}$

$y = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}$

Since $0 < r < 1$, then $r^{12} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}(\sqrt{3} - 1)$ (Shown)

73
Solution

(a) Given $S_n = kn^2 + (k+1)n$ .... (1)

Replace n by $n-1$ in (1)

$S_{n-1} = k(n-1)^2 + (k+1)(n-1)$ .... (2)

$(1)-(2):\quad u_n = S_n - S_{n-1}$

$= kn^2 + (k+1)n - [k(n-1)^2 + (k+1)(n-1)]$

$= kn^2 + (k+1)n - k(n-1)^2 - (k+1)(n-1)$

$= kn^2 + (k+1)n - kn^2 + 2kn - k - (k+1)n + k + 1$

$= 2kn + 1$ (Shown)

(b) From **(a)**: $u_n = 2nk + 1$ .... (3)

Replace n by $n-1$ in (3)

$u_{n-1} = 2k(n-1) + 1$ .... (4)

$(3)-(4):\quad u_n - u_{n-1}$

$= 2nk + 1 - [2k(n-1) + 1]$

$= 2nk + 1 - 2nk + 2k - 1$

$= 2k$ is a constant independent of n

$\therefore$ the sequence is an arithmetic sequence. (Shown)

(c) Substitute $n = 31$ into (1)

$u_{31} = 2(31)k + 1$

$= 62k + 1$

Substitute $n = 31$ into (1)

$u_6 = 2(6)k + 1$

$= 12k + 1$

Substitute $n = 1$ into (1)

$u_1 = 2(1)k + 1$

$= 2k + 1$

Given that u_{31}, u_6, u_1 are the first three terms of a geometric sequence,

i.e. $\dfrac{u_6}{u_{31}} = \dfrac{u_1}{u_6}$

$\dfrac{12k+1}{62k+1} = \dfrac{2k+1}{12k+1}$

$(12k+1)^2 = (2k+1)(62k+1)$

$144k^2 + 24k + 1 = 124k^2 + 64k + 1$

$20k^2 - 40k = 0$

$20k(k-2) = 0$

$k = 0$ (rejected $\because k \neq 0$) or $k = 2$

(d) $r = \dfrac{12k+1}{62k+1}$

When $k = 2$,

$= \dfrac{12(2)+1}{62(2)+1}$

$= \dfrac{1}{5}$

Since $|r| = \dfrac{1}{5} < 1$, the geometric series is convergent.

$u_{31} = 62k+1$

When $k = 2$, $u_{31} = 125$

Sum to infinity

$= \dfrac{u_{31}}{1-r}$

$= \dfrac{125}{1-\frac{1}{5}}$

$= 156.25$

74
Solution

(a)(i) Let $u_n = \log_2 y^n$. $\therefore\ u_{n-1} = \log_2 y^{n-1}$

$u_n - u_{n-1}$
$= \log_2 y^n - \log_2 y^{n-1}$
$= n\log_2 y - (n-1)\log_2 y$
$= \log_2 y$
Since $\log_2 y$ is a constant, $\log_2 y$ is a constant independent of n.
Thus the series is a arithmetic series. (Shown)

(b)(ii) Sum to n terms of the series
$\log_2 xy + \log_2 (xy^2) + ... + \log_2 (xy^n)$
$= \log_2 xy + \log_2 (xy^2) + ... + \log_2 (xy^n)$ ⊲ law of logarithm
$= \left(\log_2 x + \log_2 x + ... + \log_2 x\right) + \left(\log_2 y + ... \log_2 y^2 + ... + \log_2 y^n\right)$
$= n\log_2 x + \left(\log_2 y + 2\log_2 y + ... + n\log_2 y\right)$
$= n\log_2 x + \dfrac{n}{2}\left[\log_2 y + n\log_2 y\right]$

(b) The geometric series has a common ratio, $r = 3x - 2$ .... (1)
For the sum to infinity exists, $|\,r\,| < 1$ .... (2)
Substitute (1) into (2)
$\therefore\quad -1 < 3x - 2 < 1$
$1 < 3x < 3$
$\dfrac{1}{3} < x < 1$ (Shown)

Sum to infinity
$= \dfrac{\frac{3x-2}{4x}}{1-(3x-2)}$ ⊲ use formuale: $S_\infty = \dfrac{a}{1-r}$, where $a = \dfrac{3x-2}{4x}$ and $r = (3x-2)$
$= \dfrac{3x-2}{12x(1-x)}$

75
Solution

(a)(i) Let u_n be the nth term and let d be the common difference.

Given $u_5 = 10$,

$4+(5-1)d = 10$ use AP formuale: $u_n = a+(n-1)d$, where $a = 4$ and $n = 5$

$$d = \frac{3}{2}$$

Substitute $d = \frac{3}{2}$, $n = 30$ and $a = 4$ into $u_n = a+(n-1)d$

$$u_{30} = 4+29\left(\frac{3}{2}\right)$$
$$= 47.5$$

Hence, the 30th term of this arithmetic series is 47.5.

(a)(ii)
$$u_{21}+u_{22}+...+u_{50} = \frac{30}{2}(u_{21}+u_{30})$$
$$=15\left(1+20\left(\frac{3}{2}\right)+4+49\left(\frac{3}{2}\right)\right)$$
$$=1672.5$$

Alternative Method

Let S_n be the sum of the first n terms.

$$u_{21}+u_{22}+...+u_{50} = S_{50}-S_{20}$$
$$=\frac{50}{2}\left(2(4)+49\left(\frac{3}{2}\right)\right)-\frac{20}{2}\left(2(4)+19\left(\frac{3}{2}\right)\right)$$
$$=1672.5$$

(b) Given : $7+77+777+7777+\ldots$

Rewrite the series as

$$7 = 7\times10^0$$
$$7+77 = 7\times10^0+7\times10^1+7\times10^2$$
$$7+77+777 = 7\times10^0+7\times10^1+7\times10^2+7\times10^3$$
$$7+77+777+7777 = 7\times10^0+7\times10^1+7\times10^2+7\times10^3+7\times10^4$$

nth term of the series in terms of n

$$= 7\times10^0+7\times10^1+7\times10^2+...+7\times10^{n-1}$$
$$= 7(1\times10^0+1\times10^1+1\times10^2+...+1\times10^{n-1})$$
$$= 7\left[\frac{1(10^n-1)}{10-1}\right]$$

nth term of the series in terms of n is $\frac{7}{9}(10^n-1)$

Sum for the first hundred terms of the series

$$=\sum_{n=1}^{100}\frac{7}{9}(10^n-1)$$

$$=\frac{7}{9}\sum_{n=1}^{100}10^n-\frac{7}{9}\sum_{n=1}^{100}1$$

$$=\frac{7}{9}\left[\frac{10(10^{100}-1)}{10-1}\right]-\frac{7}{9}100$$

$$=\frac{70}{81}\left[(10)^{100}-91\right] \quad \text{(Shown)}$$

76
Solution

(a) Let S_n be the sum of the first n terms, a be the first term, d be the common difference, u_n be the nth term and n be the number of terms.
Given that the sum of the first ten terms of the arithmetic series is 115,
i.e. $S_{10} = 115$
$\therefore$ $5(2a + 9d) = 115$ .... (1)
Also given sixth term of the series is 13,
i.e. $u_6 = 13$
$a + 5d = 13$ .... (2)
From the GC, $a = -2, d = 3$

Sum of the 11th to 20th term
$= S_{20} - S_{13}$
$= 10[2(-2) + 19(3)] - 115$
$= 415$

(b) Given that the sum of all the terms after the tenth term is twice that of the first ten terms,
i.e. $S_\infty - S_{10} = 2S_{10}$

$$\frac{a}{1-r} - \frac{a(1-r^{10})}{1-r} = \frac{2a(1-r^{10})}{1-r} \quad \triangleleft \text{ Multiply } \frac{1-r}{a} \text{ on both sides}$$

$$1 - (1 - r^{10}) = 2(1 - r^{10})$$

$$r^{10} = \frac{2}{3}$$

$$r = \left(\frac{2}{3}\right)^{\frac{1}{10}}$$

$r = 0.96$ or $r = -0.96$ (rejected $\because r > 0$)

77
Solution

(a) Given $S_n = 6 - \dfrac{2^{n+1}}{3^{n-1}}$ .... (1)

Replace n by $n-1$ in (1)

$$S_{n-1} = 6 - \frac{2^{n+1}}{3^{n-1}} - \left(6 - \frac{2^n}{3^{n-2}}\right) \text{ .... (2)}$$

(1) − (2): nth term $= S_n - S_{n-1}$

$$= 6 - \frac{2^{n+1}}{3^{n-1}} - \left(6 - \frac{2^n}{3^{n-2}}\right)$$

$$= \frac{2^n}{3^{n-2}} - \frac{2^{n+1}}{3^{n-1}}$$

$$= \frac{2^n}{3^{n-2}}\left(1 - \frac{2}{3}\right)$$

$$= \frac{2^n}{3^{n-1}}$$

$\therefore$ nth term $= \dfrac{2^n}{3^{n-1}}$ .... (3)

Replace n by $n-1$ in (3)

$(n+1)$th term $= \dfrac{2^{n+1}}{3^n}$

$$\frac{(n+1)\text{term}}{n\text{th term}}$$

$$= \frac{\frac{2^{n+1}}{3^n}}{\frac{2^n}{3^{n-1}}}$$

$$= \left(\frac{2^{n+1}}{3^n}\right) \times \left(\frac{3^{n-1}}{2^n}\right)$$

$= \dfrac{2}{3}$ is a constant independent of n

$\therefore$ the sequence is a geometric series.

Substitute $n = 1$ into (3)

$$\therefore \text{1st term} = \frac{2^1}{3^{1-1}}$$

$$= 2$$

$\therefore$ the first term is 2 and the common ratio is $\dfrac{2}{3}$

Sum to infinity

$= \dfrac{2}{1 - \frac{2}{3}}$ $\lhd$ sum to infinity formuale: $S_\infty = \dfrac{a}{1-r}$, where $r = \dfrac{2}{3}$ and $a = 2$

$= 6$

(b) Let a be the first term, d be the common difference, T_n be the nth term and n be the number of terms.

Given that the 6th term is 15,

i.e. $T_6 = 15$

$a + 5d = 15$ .... (4)

Also given the 21st term is greater than the 15th term by 12

$T_{21} - T_{15} = 12$

$(a + 20d) - (a + 14d) = 12$

$6d = 12$

$d = 2$ .... (5)

Substitute (1) into (2)

$5 + 5(2) = 15$

$a = 5$

$$S_{30} = \frac{30}{2}[10 + 29(2)]$$
$$= 1020$$

and $$S_5 = \frac{5}{2}[10 + 4(2)]$$
$$= 45$$

$\therefore$ sum of T_6 to $T_{30} = 1020 - 45$
$= 975$

Given that the sum of first nth terms is more than the sum of the 6th to 30th term by 2046,

i.e. $\frac{n}{2}[10 + (n-1)(2)] - 975 > 2046$

$n^2 + 4n - 3021 > 0$

$(n + 57)(n - 53) > 0$

$n > 53$ $\quad$ $n < -57$ (Rejected since n is positive)

$\therefore$ the least value of n is 54

78
Solution

(a) Given AP : First term $= a$ and common difference $= d$,

GP : First term $= b$ and common ratio $= r$

The second, fourth and sixth term of the arithmetic progression are equal to the first, fourth and eighth term of the geometric progression respectively.

$\therefore \quad a + d = b$ .... (1)

$a + 3d = br$ .... (2)

$a + 5d = br^7$ .... (3)

Equating

$$(3) - (2) = (2) - (1)$$
$$br^7 - br^3 = br^3 - b \quad \text{(since } b \text{ is non-zero)}$$
$$r^7 - 2r^3 + 1 = 0 \qquad \text{(Shown)}$$

(b) Using G.C. to solve $r^7 - 2r^3 + 1 = 0$

$r = 1, r = -1.2578, r = 0.92057$

Since d is non-zero, $r \neq 1$.

$r = -1.2578$ is rejected since the geometric progression has positive terms, so r must be positive.

Hence $r = 0.921$ is the only answer.

For geometric progression to be convergent, $|r| < 1$.

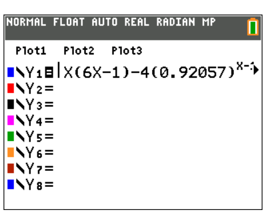
Since $r = 0.921$, hence the geometric progression is convergent.

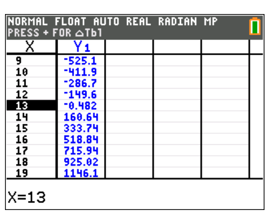
(c) Given that the difference between the sum of the first $2n$ terms of this arithmetic progression and the nth term of the geometric progression is at most $1000k$,

i.e. $\quad \left|S_{2n} - u_n\right| \le 1000k$

$$\therefore \quad \left|\frac{2n}{2}\left[2k + (2n-1)(3k)\right] - 4k(0.92057)^{n-1}\right| \le 1000k$$
$$\left|n(6n-1) - 4(0.92057)^{n-1}\right| - 1000 \le 0$$

Use GC to solve the inequality





Largest value of n is 13

79
Solution

(a) Refer to the sequence below

$$\{1\} \quad \{4,\ 7,\ 10\} \quad \{13,\ 16,\ 19,\ 22,\ 25\} \quad \{28,\ 31,\ 34,\ 37,\ 40,\ 43,\ 46\} \ \ldots$$

Note the number of terms in each bracket.

first bracket: 1 term

second bracket: 3 terms

third brackets: 5 terms

four brackets: 7 terms

From observation: 1, 3, 5. 7. ...

number of terms in each bracket follows an AP with first term 1 and common difference 2.

Number of integers in the first n sets

$=\frac{n}{2}[2(1)+(n-1)2]$

$=n^2$

(b) **Method 1**

Refer to the sequence below

$$\{1\} \quad \{4,\ 7,\ 10\} \quad \{13,\ 16,\ 19,\ 22,\ 25\} \quad \{28,\ 31,\ 34,\ 37,\ 40,\ 43,\ 46\} \ \ldots$$

Last term in 2nd set is 4th term of the AP

Last term in 3rd set is 9th term of the AP

Last term in 4th set is 16th term of the AP

In general, the last integer in the nth set is the (n^2)th term of the AP

This follows that

1, 4, 7, 10, 13, 16, ...

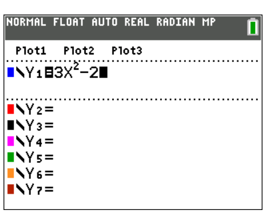
the sequence follows AP which has first term 1 and common difference 3.

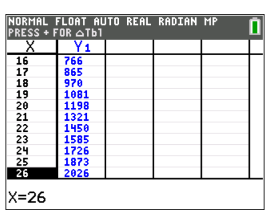
Last integer of the nth set

$=1+(n^2-1)3$

$=3n^2-2$

From GC,





The value of k that the integer 2023 occurs is in in the 26th set.

Method 2

Given that 2023 occurs in the kth set,

first term in the kth set $\le 2023 \le$ last term in the kth set

$$\left[3(k-1)^2-3\right]+3 \le 2023 \le 3k^2-2$$

From GC $\quad -24.961 \le k \le 26.961$

Since $k \in \mathbb{Z}^+$, $k=26$

(c) **Method 1**

sum of all the integers in the 5th to 10th set

$= \text{Sum of first } 10\text{th terms} - \text{Sum of first } 4\text{th terms}$

$= \frac{10^2}{2}\left[2(1)+(10^2-1)3\right] - \frac{4^2}{2}[2(1)+(4^2-1)3]$ $\triangleleft$ use sum of GP, where $a=1,\ d=3$ and number of terms $= n^2$

$= 14574$

Method 2

Refer to the sequence below

$$\{1\}\quad \{4,\ 7,\ 10\}\quad \{13,\ 16,\ 19,\ 22,\ 25\}\quad \{28,\ 31,\ 34,\ 37,\ 40,\ 43,\ 46\}\ \ldots$$

From **(b)**, Last integer of the nth set $= 3n^2 - 2$

Last term in the 10th set $= 3(10)^2 - 2 = 298$

First term in the 5th set $= 46 + 3 = 49$

It forms another AP sequence

$$49,\ 52,\ 55,\ \ldots,\ 298$$

$$298 = 49 + (m-1)3$$

$$m = 84$$

$\therefore$ there are 84 terms in this sequence.

Sum of all the integers in the 5th to 10th set

$= \frac{84}{2}(49 + 298)$

$= 14574$

80
Solution

(a) Note that the speed of the ball rolling following an A.P

Let u_n be the general term and d be the common difference

Given that at 5 seconds, the speed of the ball is 13.9 cm/s,

i.e. $u_5 = 13.9$

$v + 4d = 13.9$ .... (1) $\triangleleft$ use AP nth term formulae

Also given that at 11 seconds, the speed of the ball is 34 cm/s,

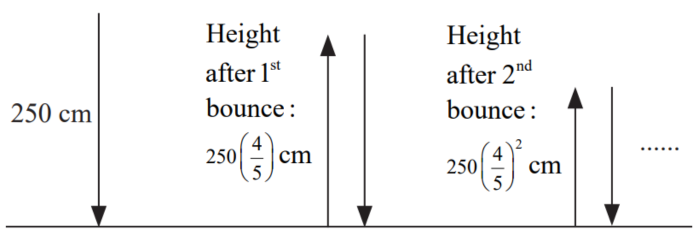
i.e. $u_{11} = 34$

$v + 10d = 34$ .... (2)

Use GC to solve (1) and (2)

$\therefore\ v = 0.5$ cm/s and $d = 3.35$ cm/s

(b)



(b)(i) $a = 250$ cm, $r = \dfrac{4}{5}$

Height after 1st bounce $= 250\left(\dfrac{4}{5}\right)$ m

Height after 2nd bounces $= 250\left(\dfrac{4}{5}\right)^2$ m

Thus, height after n th bounces $= 250\left(\dfrac{4}{5}\right)^n$ m

Total vertical distance that the ball has travelled when it reaches the highest point after the fourth bounce

$$= 250 + \left[2\left(\frac{4}{5}\right)(250) + 2\left(\frac{4}{5}\right)^2(250) + 2\left(\frac{4}{5}\right)^3(250)\right] + \left(\frac{4}{5}\right)^4(250)$$

$= 1328.4$

$= 1328$ cm (to nearest cm) (Shown)

(b)(ii) Total distance when the ball reaches the hightest point after the nth bounce

$$= 250 + 2\left(\frac{4}{5}\right)(250) + 2\left(\frac{4}{5}\right)^2 (250) + \ldots + 2\left(\frac{4}{5}\right)^{n-1} (250) + \left(\frac{4}{5}\right)^n (250)$$

$$= 250 + 2(250)\left[\frac{4}{5} + \left(\frac{4}{5}\right)^2 + \ldots + \left(\frac{4}{5}\right)^n\right] - 250\left(\frac{4}{5}\right)^n$$

$$= 250 + 500\left[\frac{\frac{4}{5}\left(1 - \left(\frac{4}{5}\right)^n\right)}{1 - \frac{4}{5}}\right] - 250\left(\frac{4}{5}\right)^n$$

$$= 250 + 2000\left[\left(1 - \left(\frac{4}{5}\right)^n\right)\right] - 250\left(\frac{4}{5}\right)^n$$

$$= 2250\left[\left(1 - \left(\frac{4}{5}\right)^n\right)\right], \quad \text{where } A = 2250 \text{ and } B = -1$$

Alternative Method

Total distance when it reaches the highest point after the nth bounce

= total downwards distance + total upwards distance

$$= \left[250 + \left(\frac{4}{5}\right)(250) + \left(\frac{4}{5}\right)^2 (250) + \ldots + \left(\frac{4}{5}\right)^{n-1} (250)\right] + \left[\left(\frac{4}{5}\right)(250) + \left(\frac{4}{5}\right)^2 (250) + \ldots \left(\frac{4}{5}\right)^n (250)\right]$$

$$= 250\left[\frac{\left(1 - \left(\frac{4}{5}\right)^n\right)}{1 - \frac{4}{5}}\right] + \left(\frac{4}{5}\right)(250)\left[\frac{\left(1 - \left(\frac{4}{5}\right)^n\right)}{1 - \frac{4}{5}}\right]$$

$$= 1250\left(1 - \left(\frac{4}{5}\right)^n\right) + 1000\left(1 - \left(\frac{4}{5}\right)^n\right)$$

$$= 2250\left(1 - \left(\frac{4}{5}\right)^n\right), \quad \text{where } A = 2250 \text{ and } B = -1$$

(b)(iii) Height of the ball reaches after nth bounce $= 250\left(\frac{4}{5}\right)^n$

When the height of the ball after nth bounce reaches less than 0.01 cm,

i.e. $250\left(\frac{4}{5}\right)^n < 0.01$

$$\left(\frac{4}{5}\right)^n < 4(10^{-5})$$

$$n \ln\left(\frac{4}{5}\right) < \ln\left[4(10^{-5})\right]$$

$$n > 45.382$$

The ball has made 46 bounces.

Total distance that the ball made 46 bounces

$$= 2250\left[\left(1-\left(\frac{4}{5}\right)^{46}\right)\right]$$

$= 2249.92$

$= 2250$ (nearest centimetre)

81
Solution

(a) $T_1 = 1000(12) = \$12000$

$T_2 = \left[1000 + 0.5(1000)\right](12) = \$18000$

$T_3 = \left[1000 + 0.5(1000) + 0.5^2(1000)\right](12) = \$21000$

$T_n = [1000 + 0.5(1000) + 0.5^2(1000) + ... + 0.5^{n-1}(1000)](12)$

$= 1000[1 + 0.5 + 0.5^2 + ... + 0.5^{n-1}]12$

$= 1000[0.5^0 + 0.5^1 + 0.5^2 + ... + 0.5^{n-1}]12$

$= 1000\left[\dfrac{1-0.5^n}{1-0.5}\right](12)$

$= 24000(1-0.5^n)$ (Shown)

Note: $0.5^0 + 0.5^1 + 0.5^2 + ... + 0.5^{n-1}$ follows geometric series with first term 0.5^0 and common ratio 0.5

(b) Total amount of money Juliana had contributed to her investment portfolio in the first N years

$= \sum_{r=1}^{N} T_r$

$= \sum_{r=1}^{N} 24000(1-0.5^r)$

$= 24000\sum_{r=1}^{N}(1-0.5^r)$

$= 24000\left(N - \sum_{r=1}^{N} 0.5^r\right)$

$= 24000N - \sum_{r=1}^{N} 0.5^r$

$= 24000N - \dfrac{0.5(1-0.5^N)}{1-0.5}$

$= 24000N - 24000(1-0.5^N)$

$= 24000(N - 1 + 0.5^N)$

82
Solution

(a) From the information:

1^{st} month : \$100

2^{nd} month : \$(100 + 110)

3^{rd} month : \$(100 + 110 + 120)

4^{th} month : \$(100 + 110 + 120 + 130)

The above series follows an arithmetic series, with first term $a = 100$ and common difference $d = 10$

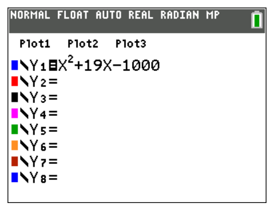
Total amount of money in the nth month

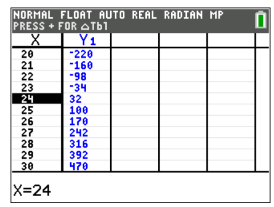
$= \dfrac{n}{2}\left[2(100) + (n-1)(10)\right]$

$= 95n + 5n^2$

For the account first become greater than \$5000,

$$\therefore \quad 95n + 5n^2 > 5000$$
$$19n + n^2 > 1000$$
$$n^2 + 19n - 1000 > 0$$





Using GC, the least number of $n = 24$

Therefore the least number of months required is 24 months.

Therefore the date is 1^{st} December 2002.

(b)

Month	Amount in the account at the beginning of month	Amount in the account at the end of month
1	100	$1.005(100)$
2	$1.005(100)+100$	$1.005^2(100)+1.005(100)$
3	$1.005^2(100)+1.005(100)+100$	$1.005^3(100)+1.005^2(100)+1.005(100)$
...	...	...
n		$1.005^n(100)+1.005^{n-1}(100)+...+1.005^3(100)$ $+1.005(100)^2+1.005(100)$

Total amount in the account at the end of nth month

$= 1.005^n(100) + 1.005^{n-1}(100) + ... + 1.005^2(100) + 1.005(100)$

$= 100\left[1.005^n + 1.005^{n-1} + ... + 1.005^2 + 1.005\right]$

$= 100\left[\dfrac{1.005(1.005^n - 1)}{1.005 - 1}\right]$

$= 20100(1.005^n - 1)$

$\therefore$ the expression for the amount in Mr B's account on the last day of the nth month is $20100(1.005^n - 1)$.

For the amount to exceed \$5000,

$$20100(1.005^n - 1) > 5000$$

$$1.005^n > 1.24875$$

$$n > \frac{\ln(1.24875)}{\ln(1.005)}$$

$$n > 44.5$$

Therefore the least number of months required is 45 months which is 3 years and 9 months.
Mr B's account first exceeds \$5000 at the start of first day of September 2004.

Therefore the month required is 1st September in 2004.

(c) Let $r\%$ be the interest rate per month.

Total amount of money in the account at the end of 35 months

$$= 100\left[\left(1+\frac{r}{100}\right)^{35} + \left(1+\frac{r}{100}\right)^{34} + \ldots + \left(1+\frac{r}{100}\right)^{2} + \left(1+\frac{r}{100}\right)\right]$$

$$= 100\left[\frac{\left(1+\frac{r}{100}\right)\left(\left(1+\frac{r}{100}\right)^{35} - 1\right)}{\left(1+\frac{r}{100}\right) - 1}\right]$$

Total amount of money in the account at the end of 35 months $+ 100$ deposit on the first day of 36th month > 5000

$$\therefore\ 100\left[\frac{\left(1+\frac{r}{100}\right)\left(\left(1+\frac{r}{100}\right)^{35} - 1\right)}{\left(1+\frac{r}{100}\right) - 1}\right] + 100 > 5000$$

$$\Rightarrow 100\left[\frac{\left(1+\frac{r}{100}\right)\left(\left(1+\frac{r}{100}\right)^{35} - 1\right)}{\left(1+\frac{r}{100}\right) - 1}\right] + 100 - 5000 > 0$$

Using GC (graphical method), $r = 1.80$ (correct to 3 sig fig)

Therefore the interest rate per month is 1.80%.

Note

"the \$5000 on 2 December 2003" means it includes the \$100 deposited on 1 December 2003. This also means 35 months has lapsed.

83
Solution

(a) Total distance covered in 1st cycle $= 100+130+130+170+170 = 700$

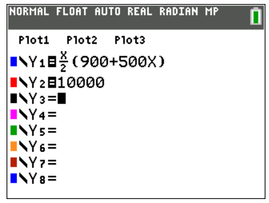
Total distance covered in 2nd cycle $= 200+230+230+270+270 = 1200$

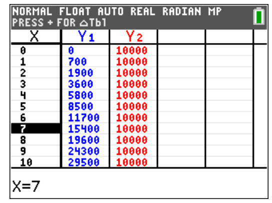
Total distance covered in 3rd cycle $= 300+330+330+370+370 = 1700$

From above, the total distance covered follows an arithmetic progression, with first term $a = 700$ and common difference $d = 500$

Let S_n be total distance that the train covered in nth cycle.

$$S_n \geq 10000$$
$$\frac{n}{2}\left[2(700)+(n-1)(500)\right] \geq 10000$$
$$\frac{n}{2}(900+500n) \geq 10000$$





From GC,

$n \leq -7.2883$ or $n \geq 5.4883$ (rejected since n is an integer)

The train has completed 5 cycles

$$S_5 = \frac{5}{2}\left[2(700)+4(500)\right]$$
$$= 8500$$

The train broke down after 1500 km in the 6th cycle.

The train will run 600 km on the first day of the 6th cycle.

The train will break down on the 3rd day of the 6th cycle.

(b) Odd days:

On Mon the train runs 100 km

On Wed the train runs $0.92(1.2)(100)$ km

On Fri the train runs $0.92^2(1.2)^2(100)$ km

Let O_n be the sum of distance the train runs on n odd numbered days.

$$O_n = 100+0.92(1.2)(100)+0.92^2(1.2)^2(100)+\ldots+0.92^{n-1}(1.2)^{n-1}(100)$$
$$= \frac{100(1.104^n-1)}{1.104-1}$$
$$= 961.54(1.104^n-1)$$

Even days:

On Tue the train runs $(1.2)100$ km

On Thur the train runs $0.92(1.2)(100)$ km

On Fri the train runs $0.92^2(1.2)^2(100)$ km

Let E_n be the sum of distance the train runs on n even numbered days

$$E_n = (1.2)100+0.92(1.2)^2(100)+0.92^2(1.2)^3(100)+\ldots+0.92^{n-1}(1.2)^n(100)$$
$$= \frac{120(1.104^n-1)}{1.104-1}$$
$$= 1153.8(1.104^n-1)$$

Let S_n be the total distance covered by n days.

$$S_{2n} = O_n + E_n$$
$$= 961.54(1.104^n - 1) + 1153.8(1.104^n - 1)$$
$$= 2115.4(1.104^n - 1)$$

(c)

$$S_{2n} > 10000$$
$$2115.4(1.104^n - 1) > 10000$$
$$1.104^n > 5.7275$$
$$n > 17.640$$

Therefore, the train must complete at least 36 days of travelling.

84
Solution

(a) Given $u_n = (1+\alpha)u_{n-1}$ .... (1)

$$\frac{u_n}{u_{n-1}} = (1+\alpha)$$

Since $\frac{u_{n+1}}{u_n} = 1+\alpha =$ constant (since α is a constant)

$\therefore \{u_n\}$ is a geometric progression.

(b) Substitute $n = 2$ into (1)

$$\begin{aligned} u_2 &= (1+\alpha)u_{2-1} \\ &= (1+\alpha)u_1 \\ &= (1+\alpha)2^m \end{aligned}$$

Substitute $n = 3$ into (1)

$$\begin{aligned} u_3 &= (1+\alpha)u_{3-1} \\ &= (1+\alpha)u_2 \\ &= (1+\alpha)\times(1+\alpha)2^m \\ &= (1+\alpha)^2 2^m \end{aligned}$$

Given that over the second and third days, a total of $\frac{91}{36}(2^m)$ units were processed,

i.e. $u_2 + u_3 = \frac{91}{36}(2^m)$

$$\begin{aligned} (1+\alpha)2^m + (1+\alpha)^2 2^m &= \frac{91}{36}(2^m) \\ (1+\alpha) + (1+2\alpha+\alpha^2) &= \frac{91}{36} \\ \alpha^2 + 3\alpha + 2 &= \frac{91}{36} \\ \alpha^2 + 3\alpha - \frac{19}{36} &= 0 \\ 36\alpha^2 + 108\alpha - 19 &= 0 \end{aligned}$$

Using GC, $\alpha = \frac{1}{6}$ or $\alpha = -\frac{19}{6}$ (rejected since $p > 0$)

Hence $\alpha = \frac{1}{6}$

(c) Since common ratio $r = \frac{7}{6} > 1$, sum to infinity does not exist.

As $n \to \infty$, total amount of data S_n diverges.

Hence there is no limit to total the amount of data that the centre can handle.

(d) Substitute the values of n into the piecewise function below

$$v_r = \begin{cases} u_r & \text{for } 1 \le r \le 4, \\ v_{r-1} - 25 & \text{for } r \ge 5. \end{cases}$$

When $n = 1$ $\quad v_1 = u_1 = 2^m$

When $n = 2$ $\quad v_2 = u_2 = 2^m\left(\frac{5}{4}\right)$

When $n = 3$ $\quad v_3 = u_3 = 2^m\left(\frac{5}{4}\right)^2$

When $n = 4$ $\quad v_4 = u_4 = 2^m\left(\frac{5}{4}\right)^3$

When $n = 5$ $\quad v_4 = u_4 = 2^m\left(\frac{5}{4}\right)^3 - 25$

When $n = 6$ $\quad v_4 = u_4 = 2^m\left(\frac{5}{4}\right)^3 - 25(2)$

When $n = 7$ $\quad v_4 = u_4 = 2^m\left(\frac{5}{4}\right)^3 - 25(3)$

.....

For $r \ge 4,$ the sequence $\{v_r\}$ follows an arithmetic progression, with first term $2^m\left(\frac{5}{4}\right)^3$ and common difference -25

From v_4 to v_r, no. of terms $= r - 4 + 1 = r - 3$

$$\therefore \quad v_r = 2^m\left(\frac{5}{4}\right)^3 + \left[(r-3)-1\right](-25)$$

$$= \frac{125}{64}(2^m) - 25(r-4)$$

$$= 125(2^{m-6}) - 25r + 100$$

$$= 25\left[5(2^{m-6}) - r + 4\right], \; r \ge 4 \text{ .... (2) (Shown)}$$

(e) Substitute $n=4$ into (2): $v_4=\frac{125}{64}(2^m)$

Substitute $n=15$ into (2): $v_{15}=25\left[5(2^{m-6})-15+4\right]$

Number of terms for $v_4+...+v_{15}=12$

The series $v_4+...+v_{15}$ is an arithmetic progression, with first term v_4, last term v_{15} and number of terms 12

Total amount of data processed from the 1st day to the 15th day, using revised operating procedure

$$=v_1+v_2+v_3+v_4+...+v_{15}$$

$$\left(v_1+v_2+v_3\right)+\left(v_4+...+v_{15}\right)$$

$$=\left[(2^m)+\frac{5}{4}(2^m)+\frac{25}{16}(2^m)\right]+\left[v_4+......+v_{15}\right]$$

$$=\frac{61}{16}(2^m)+\frac{12}{2}(v_4+v_{15})$$

$$=\frac{61}{16}(2^m)+6\left[\frac{125}{64}(2^m)+\left(25\left[5(2^{m-6})-15+4\right]\right)\right]$$

$$=\frac{61}{16}(2^m)+6\left[\frac{125}{64}(2^m)+\left(\frac{125}{64}(2^m)-275\right)\right]$$

$$=\frac{61}{16}(2^m)+6\left[\frac{125}{32}(2^m)-275\right]$$

$$=\frac{109}{4}(2^m)-1650 \quad \text{where } s=\frac{109}{4},\ t=-1650$$

(f) Based on revised operating procedure, amount of data processed after a certain number of days will become negative due to AP with negative common difference. Since it is not possible to process negative amount of data, it is not meaningful to compute such data in the long run.

85
Solution

(a) Hoist A

Distance covered at 1st pull $= 45$

Distance covered at 2nd pull $= 41.8$

Distance covered at 3rd pull $= 40.2$

Observed that the sequence follows AP with first term 45 and common difference -1.6

Distance covered at the nth pull

$= 45 + (n-1)(-1.6)$

$= 46.6 - 1.6n$

Consider $46.6 - 1.6n \geq 0$

$n \leq 29.125$

Hence number of pulls needed to achieve maximum total height is 29.

Maximum total height

$= \dfrac{29}{2}\left[2(45) + (29-1)(-1.6)\right]$ ◁ use sum of first n term of AP, where $a = 45$, $d = -1.6$ and $n = 29$

$= 655.4$ cm

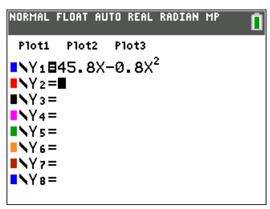
Alternative Method

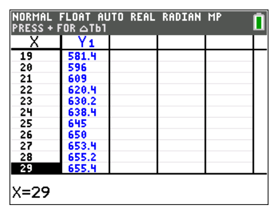
Let S_n be the total height of the first n pulls.

$S_n = \dfrac{n}{2}\left[2(45) + (n-1)(-1.6)\right]$

$= 45.8n - 0.8n^2$

Using GC,





Hence the number of pulls needed to achieve maximum total height is 29, and the maximum total height covered is 655.4 cm.

(b) Since $r = 0.95 < 1$, sum to infinity of G.P. exists.

Maximum total height that Hoist B will raise the load

$= \dfrac{45}{1-0.95}$

$= 900$ cm

Note

theoretical maximum refers to sum to infinity

(c)(i)

	Total height reached
Before 2nd pull	$0.98(45)$
Before 3rd pull	$0.98\left[0.98(45)+45\right]$
Before 4th pull	$0.98\left[0.98^2(45)+0.98(45)+45\right]=0.98^3(45)+0.98^2(45)+0.98(45)$

Total height reached before 4th pull

$=0.98^3(45)+0.98^2(45)+0.98(45)$

$=\dfrac{0.98(45)(1-0.98^3)}{1-0.98}$

$=129.67164$

$=130\text{ cm}\quad(3\text{ s.f.})\quad(\text{Shown})$

(c)(ii)

	Total height reached
Before 2nd pull	$0.98(45)$
Before 3rd pull	$0.98\left[0.98(45)+45\right]$
Before 4th pull	$0.98\left[0.98^2(45)+0.98(45)+45\right]=0.98^3(45)+0.98^2(45)+0.98(45)$

From the above series, we can derive the total height reached before $(n+1)^{\text{th}}$ pull

Before $(n+1)^{\text{th}}$ pull, total height reached

$=0.98^n(45)+0.98^{n-1}(45)+\ldots+0.98^2(45)+0.98(45)$

$=0.98^1(45)+0.98^2(45)+\ldots+0.98^{n-1}(45)+0.98^n(45)$

$=\dfrac{0.98(45)(1-0.98^n)}{1-0.98}$ $\triangleleft$ use sum of first nth terms of GP with first term 0.98(0.45), common ratio 0.98

$=2205-2250(0.98)^{n+1}$

(c)(iii) From **(c)(ii)**, total height reached by load using hoist C $=2205-2250(0.98)^{n+1}$

As $n\to\infty, (0.98)^{n+1}\to 0.$

Hence maximum total height $\to 2205.$

The maximum total height reached by load using hoist C will approach 2025 cm.

Therefore the hoist C cannot be used to lift the load up to height of 2500 cm.